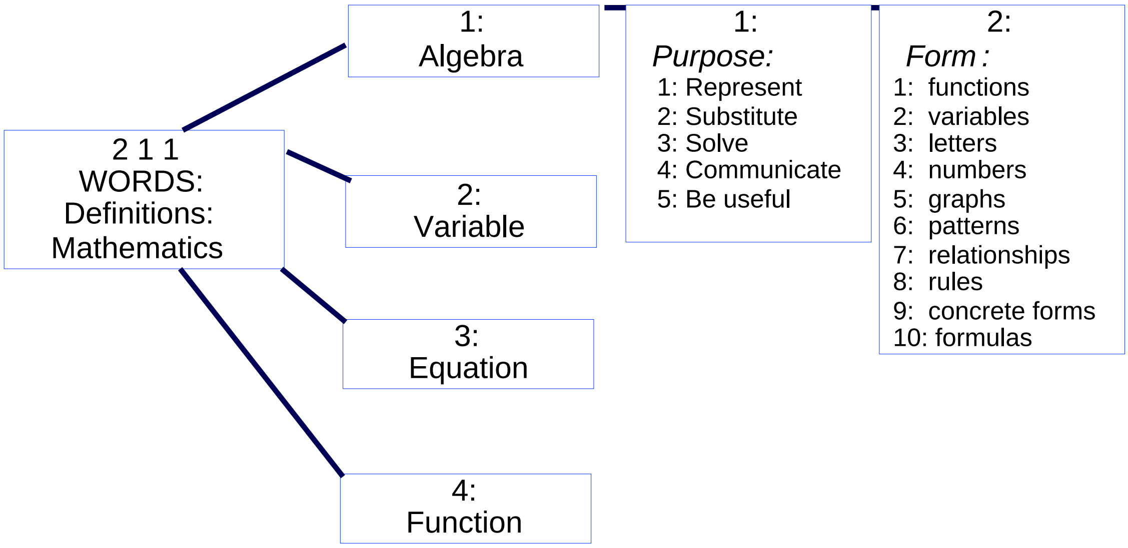
Six
Thinking about Algebra

Definitions of Algebra

The mathematical definitions elicited within the study centred upon four elements - definitions of algebra itself, and of three central constituents of algebra - variable, equation and function. The data gave rise to an analytical feature common to all four definitions - the distinction between **form** and **purpose** (akin to Piaget's **structure** and **function**). Responses identified with **form** were those associated with attempts to describe the nature of algebra in terms of its constituent parts. These included "a group of letters that may stand in place of numbers" (S3: Jane), "algebra is many numbers but expressed as letters" (S1: Andrea) and "algebra is an equation or rule" (S4: Stephen). Each reflects a perception of what algebra *is*, as opposed to what algebra *does*. This latter is associated here with a conception of **purpose**, and is reflected in definitions such as "algebra is finding or solving or using unknowns to get a correct solution, it is like putting one number in then getting another number out to help solve the problem" (S1: Andrea). Within both these distinctions may be found a range of perspectives which make up the sub-categories depicted in Figure 6.1.

Figure 6.1: Definitions of mathematics



The form or structure of algebra involved the identification of ten different descriptors. **Functions** were mentioned explicitly only by the researcher ("The principal concepts of algebra are functions and variables - functions defining a relationship between variables which can be expressed in various representations" (T1: SMA)) and two of the student teachers. Both of these were in a subsidiary context to the concept of variable (A3: "Through the use of their interpretation and analysis skills of graphical presentations, students are in the position to gain insights into the dynamism of functions and variables," and A4: "students can see how the letters can represent a function".) The emphasis placed upon the concept of **variable** in their preceding course related to algebra teaching and learning was clear in the responses of the Group A preservice teachers. All saw it as important that students "develop an understanding of variable and the relationships between variables in Algebra" (A1) and "for students to be successful in this introductory stage of algebra learning, and in fact, all other stages, they must develop an understanding of what a variable is" (A3). Each of the preservice teachers from Group A who made an attempt to define algebra included specific and explicit mention of the central role of understanding of variable. None of the secondary students made such explicit reference to either functions or variables in their definitions of algebra.

For the secondary students, the nature of algebra was defined by **letters** and **numbers**, with less frequent references to **rules, formulas** and **graphs**. An early interview with Jane (S3) as a Year 10 student studying Advanced level mathematics revealed a shallow understanding of this nature:

Interviewer: .	..This person, towards the end of the lesson, nudges you again … and says, "What's algebra?" We've had this one before, but how would you explain it to someone who didn't know?
Jane:	Letters?
Interviewer:	Then he says, "Letters?" Yeah, that's what we do in English.
Jane:	Um … numbers and letters?
Interviewer:	Okay…
Jane:	Um … a group of letters that mean something, equal something?
Interviewer:	Okay, a group of letters that mean something, equal something … equal what? Equal numbers?
Jane:	Yeah.
Interviewer:	Okay, so a letter like "a" can stand for a number?
Jane:	Yeah.
Interviewer:	Okay, can it stand for more than one number?
Jane:	Yes.
Interviewer:	So it can stand for what? Two numbers?
Jane:	It can stand for … I don't know … anything.
Interviewer:	Anything?
Jane:	Any numbers.

This transcript is revealing of the tacit nature of knowledge about algebra. It is clearly disjointed and difficult to articulate for the student and yet, upon probing, is essentially sound in its foundations. The important concept of variable as representing a range of values rather than simply a “placeholder” is present here, suggested at the end of the transcript. It was as a consequence of this recognition of the tacit nature of students’ knowledge about algebra that alternative ways of gathering research data were explored, leading to the image-based methods developed later in the study.

Table 6.1

Forms of Algebra

Forms	function	variable	letters	numbers	graphs	patterns	relations	rules	concrete	formulas
A1		√				√	√	√	√	√
A3	√	√		√	√	√		√	√	√
A4	√	√		√		√		√	√	√
A5		√						√	√	√
A6		√						√	√	√
S1			√	√				√		√
S2					√			√		√
S3			√	√						
S4			√	√	√	√		√		√
S5			√	√						
S6									√	
T1	√	√					√			

Table 6.1 presents a summary of responses to the nature of algebra which reflect a consideration of its constituent components. Several subgroups may be discerned from these responses. Patrick (S6 - Year 7) might best be termed *pre-algebraic*, his only experience of algebra at this stage being informally, through concrete models. Jane (S3 - Year 10) and Tony (S5 - Year 8) are both limited to a *unistructural* understanding of algebra, as "letters standing for numbers". It would appear that entry into senior mathematical study brings with it a broader conception which commonly includes recognition of the roles of symbolism (equations and formulas) and relationships in defining the nature of algebra. Such an understanding, however, appears at best *multistructural*, even among the preservice teachers, as their understanding is made up of a collection of largely unrelated parts. Understanding of the relationships between the various components of algebra - the letters, the numbers, the graphs, the rules and formulas, even the place of functions and variables - appeared to be largely absent among both students and preservice teachers, at least as revealed by the verbal definitions offered.

Five dimensions of algebraic *purpose* were also identified from the data. Most common of these was the perceived function of **solving**, found among all the older students (Andrea (S1), Ben (S2), Jane (S3) and Stephen (S4)) and two student teachers (A5 and A6). These definitions included "algebra is a form of mathematics which helps us in ways to solve mathematical problems" (Ben), "we use algebra to find out numbers that we do not know" (Tony), and "I believe algebra to be a mathematical expression for finding unknowns" (A5).

Another common perception of the purpose of algebra was its **representative** function - most commonly as letters standing for numbers (already mentioned above), but with other references to **graphs, patterns** and **concrete forms** as serving representational roles. Among the secondary students, there was evidence of a shift over time, from a **static** view of algebra (aligned with its representative function) to a more **active** view (associated especially with **solving** problems). Jane's early definition of algebra, captured in the interview transcript above, is persistent over time ("letters and numbers", "letters that stand for numbers" and "algebra here means anything with letters in it") until finally, it includes a new component: "Algebra is a group of numbers and letters to solve equations". This added dimension of purpose in addition to form appears to suggest some change in perception over the period of the study.

Jane's passive view of algebra may be contrasted with a more active view, such as that consistently displayed by Stephen (S4). From his earliest definition ("Algebra is a topic of maths which has letters and numbers. Solve these equations [sic] and simplifying them is the main target. Algebra is no set form and can be described on graphs or number lines or other ways".) Stephen clearly displayed a more comprehensive view of algebra than that of Jane, a view that was similarly persistent over the period of the study: "Algebra is a way of understanding a[n] equation with an unknown value used to describe the pattern it makes. It is useful to know how to deal with algebra so you can know how to understand statements without knowing its value". Compare this early definition with later ones: "Algebra is an equation or rule which is a guideline to answering certain types of equations which involve pronumerals" and "Algebra is a way of describing a graph or equation with pronumerals and numerals".

Stephen's quite comprehensive definitions may be compared with those of his peer, Ben (S2). While Stephen was attempting the higher level Three Unit mathematics course for the Higher School Certificate, Ben was attempting the Two Unit course. For Ben, "Algebra is a form of mathematics which helps us in ways to solve mathematical problems. It's all done with formulas and visual aids - graphs, plotting in the number plane, curves and different equations that you can graph". Much later, this had been refined to "Algebra is solving algebraic problems (graphically, algebraically) where there is always an unknown". For Ben, algebra is about solving, whereas Stephen's view included a clear descriptive function as well.

Andrea (S1), too, shared this perception. Developing from her early representative depiction, "algebra is many numbers but expressed as letters", her view of algebra grew over time to become far more active: "Algebra is finding or solving or using unknowns to get a correct solution". Like Stephen, Andrea was attempting the higher level Three Unit course in Year 11 at the time of the study. Their definitions appeared to be revealing of a broader conception of algebra than that of Ben.

As a Year 8 student, Tony (S5) might be expected to display quite limited understanding of the nature and role of algebra. It is interesting, then, to note that even his earliest attempt - "Algebra is where you use letters to substitute for numbers" - a clear element of action was evident, and such a view was labelled **substitution** to distinguish it from the passive **representation** form displayed by others, such as Jane. A much later definition showed the twin elements of representation and solution which had also categorised the developments of the older students:

> Algebra is the use of letters in mathematics as pronumerals for numbers that we do not know. We use algebra to find out numbers that we don't know. An equation is using algebra to find out the value of a pronumeral.

Although Tony was graded at this time as a middle ability student at his school (placed in the third of five graded mathematics classes), his responses seem indicative of a higher ability range than this. (Towards the end of the study, Tony had actually been promoted two class levels, suggesting that student understanding of mathematical concepts may be a useful indicator of mathematical capabilities. At the same time, the ability to express oneself verbally in a clear and articulate way may also be a relevant indicator in this context, possibly independent of mathematical ability.)

This function associated with a process of "solving" was commonly linked to another, **being useful**, in which explicit recognition was made of algebra serving some helpful role. Although the *purposes* of algebra were categorised in a variety of ways already discussed, few mentioned any *application* of algebra beyond its own ends. In other words, for most participants, the purpose of algebra is to solve algebraic problems. Of the students, Andrea noted that "we do algebra to make life easier - without algebra we wouldn't have half the stuff we do now, or maths", while Tony suggested that "it helps us in everyday life when adding stuff up at the supermarket (a basic example) and all kinds of other ways" (although he may well have been referring here to mathematics in general). Even the preservice teachers, at the end of their formal studies of mathematics and pedagogy, were able to offer only that “algebra is a highly useful area of mathematics that is a crucial foundation to much of the mathematics that High School students learn” (A6) and “algebra is important to pupils learning of mathematics. It is the basis of the maths pupils will do in later years”. (A2).

Only one participant (A3) offered an alternative view of algebra, as a means of **communicating**:

> Students need to view algebra as a language, a way of saying or communicating a rule in an abbreviated form. The use of symbolism, the abbreviated form, incorporates the idea of generalisation.

This view hints at the potential of algebra as a mediating tool which supports thinking and enables higher cognitive functioning in the same way as language. It was evident nowhere else in the data, however, and is clearly not a general perception of algebra among the participants.

The definitions of *equation, variable* and *function* were similarly analysed into statements related to **form** and **purpose**. The sub-categories for each are summarised below, along with those participants from whom these categories were derived.

Thinking about Equations

Equation is...	
(1) Purpose	*(2) Form*
• To represent:	• manipulations (A1, A4)
- equality (S3, S4)	• representations:
- relationship (S5)	- graphical (A1, S4)
• To solve (S1)	- symbolic (A4, S3, S4)
	• variables (A1, S1, S5)

The familiar identification of equations with the manipulation of symbols to produce an answer was explicitly found only among the student teachers. For the students, an equation was more likely to be defined in terms of graphs and symbols arranged around an equality. While the notion of equality and the purpose of deriving an "answer" were generally recognised properties, the means by which this answer might be derived was notably absent from consideration. Once again, some responses among the students were clearly **unistructural** ("a set of numbers and letters that equal another set" (S3: Jane) and "when you make a rule or relationship between numbers and pronumerals" (S5: Tony), while others indicated thinking which was **multistructural** ("An equation is a set of numbers which might contain variables - when you put one number in, another number will come out. e.g. x + 1 = y (two variables)" (S1: Andrea) and "an equation is a statement which describes a line or curve which can be drawn on a graph. It includes a group of numbers with letters and their values. You try and get the letters (usually x and y values) to equal a number" (S4: Stephen).

Comparison of the two responses classified as unistructural reveals that, while both focus upon a single property, they are nonetheless quite distinct in nature. A similar finding occurs with the second pair of examples. In both cases, the second response illustrates a higher cognitive level than the first. Consider, for example, the two responses classified as unistructural. The object of focus for the first is a static notion of equality. It views the equation as a totality and appears to illustrate what van Hiele terms a "visual" response. The second response, however, takes as its focus the establishment of a relationship, an active perception quite different to the first. Similarly, the second multistructural response is far more complex than the first. These examples appear to support the recent developments in the SOLO taxonomy which suggest that, within a single mode of thinking (in this case, concrete-symbolic), there may be found several cycles of increasing complexity, rather than the single unistructural-multistructural-relational cycle originally proposed (Pegg, 1992).

Perhaps even more illuminating is the distinction drawn by van Hiele between what he terms the **symbol** and **signal characters** of cognitive objects such as geometric figures or, as in this case, equations and algebraic symbols (van Hiele, 1986, pp. 60-61, p. 168). Initially, he proposes, an object is recognised by its **symbol character** (van Hiele, 1986):

> Many symbols begin their existence with an image in which the observed properties and relations are temporarily projected. However, after the explication of those properties and relations by an analysis or discussion, the symbol loses the character of image, acquires a verbal content, and thus becomes more useful for operations of thinking. (p. 61)

Such a view is likely to be visual in nature - global, wholistic, intuitive. Gradually, however, van Hiele submits that the symbol acquires new properties - "the symbols act as signals" (van Hiele, 1986, p. 62). As such, they influence the thinking of the individual in a direct and deliberate way. As signals, they will trigger a cognitive reaction, which may be a recognition of a complex of properties and relationships or may even be a *signal to act* within a specific context.

The example of equation is an interesting case in point. When an individual views an equation, what is actually seen? At the lowest level (in a *prealgebraic* sense) an equation is simply a jumble of letters and numbers (similar to Jane's response above). Having acquired a symbol character through early study of algebra, it is recognised as defining a relationship of equality (Tony's response). (This does not imply that Jane was operating at a prealgebraic level, only that her response might be seen as illustrative of such a level).

If the second pair of responses are considered, both illustrate recognition of signal characteristics of equations. Much of the training in early algebra is intended to produce a particular signal response in students - an automatic triggering of the signal to act in a predetermined sequence which will eventually result in a solution. Both multistructural responses above display this signal nature - they recognise that an equation is an object to be used to produce a result, an answer. Stephen's response, however, appears to go one step further. His view of equation triggers not only the signal to act, but also the recognition of other representations of equation which are indicative of a richer network of relationships.

What an individual sees when confronted by an equation, a function, a graph or any of the symbolic objects associated with algebra must be recognised as a central issue in the present study. Further insights into such thinking are likely to arise from consideration of responses to visual images of algebra, and it seems likely that the use of computer tools must be considered in the context of such visual thinking.

Thinking about Variables

Variable is...	
(1) Purpose	*(2) Form*
• To represent:	• pronumeral (A1,A3,A4,S1,S5)
- unknown value (S1, S4)	• patterns (A1, A4, A5)
- numerical value (A2, A3, S4, S5, S6)	• rule (A2, S4)
- range of values (A1, A3, A4, A5, A6, S4, S6)	• dynamic (A2, A3, A5, S4, S5)
	• letter (A3, S4, S6)
- lines/curves (S4)	
• To solve (S4)	
• To simplify (S6)	

Understanding of the concept of variable has been a common focus for studies associated with the learning of algebra (for example, Quinlan, 1992) including learning within a computer-based context (Boers, 1992). In the latter case, it was found that students with access to computer algebra software were more likely to think of variables as representing a range of values than as a single placeholder for an unknown object. Such was found to be the case in the present study, where student definitions of variable tended to be active, process-oriented conceptions: Stephen (S4), for example, thought of variable early in the study as “a letter used in algebra which is used to describe certain lines or curves or to use as an unanswerable value. Most variables are able to be used to find answers to... They are mostly used in rules of finding answers". Later, however, a variable was "how you change the answer, by using different numbers". Andrea (S1), too, described variables in terms of multiple unknowns: "a symbol representing an unknown anything - usually numbers, or a degree (angle)".

Of considerable interest were the responses of the two junior secondary students, Patrick (S6) and Tony (S5). Tony had not encountered variables formally outside his use of the computer-based instructional modules developed for the study. His active perception of variable as "a changing amount of numbers or pronumerals in a sum, equation or whatever" may be attributed to his computer-based learning context. Patrick, even more so, had studied no algebra, and had worked through the *Beginning Algebra* modules and used the *Concrete Algebra* modes available. His comments display a firm and clear understanding of the concept of variable within the symbolic context of algebra:

> My theory about variables is that by replacing words with single letters it makes maths easier for those who have troubles with long words. The letters stand for amounts of things and numbers ... [A variable is] replacing amounts of substances or numbers with single letters. It can stand for either one or many numbers.

Later, when describing the meaning of given concrete shapes to which had been assigned letters representing their areas, Patrick noted:

> 'm' stands for any shape or form which covers five squares, m = 5.
> 's' stands for any shape which covers six squares, s = 6.
> 'a' stands for any shape which covers two squares, a = 2.
> At the moment, m = 5 but this is not a permanent fixture as it is only this at the time. And the same rules apply for other letters.

While it is hardly surprising that the student teachers displayed consistent and versatile understanding of the concept of variable (since it had been an area of particular focus in their previous studies), the depth of understanding of the concept displayed by the secondary students appears to derive to a great extent from the technology-rich algebra learning environment which they had shared, within which the attainment of an active process conception of variable had been an explicit priority.

Thinking about Functions

Function is...	
(1) Purpose	*(2) Form*
• To represent:	• graph (A1, A3, A5, S3)
- unique value (A1- 6)	• rule (A1, A3, A4, A6, S4)
- non-unique value (S4)	• input/output machine (A1, A6, S3, S4)
- action (A1, S3)	• domain/range (A1, A2, A3, A5, A6)
• To solve (A4, S1, S4, S6)	• table of values (A3)
	• equation (A5, S1, S4)
	• unknowns (A4, A5)
	• patterns (A5, S4)
	• set of numbers (A5, S3)
	• relationship (S4, S6)

In addition to the concept of variable, the other area of particular focus in the construction of the computer-based learning environment for the study was an understanding of **function**. The emphasis within the program was upon building a **versatile** conception of function, within which students would have access to a relatively rich cognitive repertoire when considering functions.

The preservice teachers displayed a thorough knowledge and understanding of this concept. All acknowledged the formal requirement for uniqueness which, although specifically mentioned within the instructional modules of the program, did not appear among the definitions of the secondary students. The latter were more likely to associate function with **solving**, arising from a perception that functions and equations were essentially the same. This confusion is evident in Andrea's definition:

> A function is a y value of an equation but not the y value when the equation equals zero. You can simplify a general form equation, however, you cannot do this to a function even though the numbers are exactly the same. e.g. f(x) = 3x^2+6x+9 is a function.

This confusion was more pronounced for Jane, who really did not know what a function was, but suspected that it was probably very similar to an equation:

Interviewer:	Well ... in your way of thinking, would that be a ... is that what you would think of as a function?
Jane:	[Long pause] No. I don't know. Um...
Interviewer:	So you're not sure?
Jane:	No.
Interviewer:	That's fine. Do you reckon they would mean the same thing?
Jane:	Yeah.
Interviewer:	So they're both really like equations?
Jane:	Yeah.

It seems likely that Jane was grasping at straws at the end of this interview, trying to escape from a difficult situation where she was being questioned about something which she really did not know. While she was familiar with the term from her mathematics classes, she really showed no understanding of its nature. She did, however, identify function, not only with equation, but with ordered pairs and with plotting sets of points onto a graph, suggesting that her mental image of function was perhaps richer than her verbal definition.

Stephen also displayed a diverse conception of function, which included number patterns, input/output machine images and the association with a rule or relationship. His understanding of function developed significantly throughout the period of the study. His early thinking was unistructural (although clearly active): "a function is [when] a value of one is determined upon a value of another". His response to the symbol **$f(x)$** was "the values that x can be in the equation written after it ... and you put numbers for the x-values, and use them to work out what that equals". Stephen was quite definite about the distinction between functions and equations - there was none.

Interviewer:	Alright, last question. This is the easy one. What's the difference, if any ... is there any difference between functions and equations?
Stephen:	No.
Interviewer:	Right. So you gave me an example of a function before which was f of 1 equals... or f of x equals x cubed minus whatever. Give me an example of an equation.
Stephen:	x squared minus 6x plus three ... equals zero.
Interviewer:	Alright, alright, so what's the important thing about an equation?
Stephen:	It equals zero, and it also has to have a number ... at the end to show where it crosses the y axis, or whatever.
Interviewer:	Alright, so, say you've got ... y equals 3x plus 1. Is that an equation? Is it a function?
Stephen:	Yes, it can be.

Interviewer:	Good. So it doesn't have to have the 'f(x)' part to make it a function? Which part would you say was the 'function part', or is the whole thing the function? y equals 3x plus one.
Stephen:	Uh, the function is ... x ... the part after the y.
Interviewer:	It is what it equals?
Stephen:	Mm.
Interviewer:	What about something like 'x plus y equals two'? Is that an equation?
Stephen:	Mm.
Interviewer:	Why?
Stephen:	Because it's got both x and y values, and it equals a number. But in a function ... it's ... the x value plus a number ... with y is the function out the front.

This interview is revealing of the fragility of understanding of basic algebraic concepts, even by students considered quite mathematically capable. Even the most common of algebraic entities, the equation, appears to present a minefield of uncertainty for students. The transcript suggests, too, that the physical arrangement of algebraic forms plays a very important part in student perceptions - Stephen's insistence that a function requires a part "out the front" is significant, particularly within a computer-based context where, for example, many graph plotters require functions to be entered in a specific format (usually "y =" or even "f(x) ="). The *HyperCard* plotter and table of values utilities developed in response to these interviews deliberately allowed functions to be entered with or without a "y =" prefix, in order to study student preferences and any potential effects upon student thinking about functions.

The interview continued, attempting to further tease out Stephen's thinking about functions and equations:

Interviewer:	... So what about something like... sine of x? Now is that a function?
Stephen:	Yeah ... mm... [unsure]
Interviewer:	Wouldn't like to put a hundred dollars on it?
Stephen:	No.
Interviewer:	Alright, is it an equation?
Stephen:	Yes.
Interviewer:	It is?
Stephen:	Just sin(x)?
Interviewer:	Sin(x).
Stephen:	No.
Interviewer:	So what would it need to be an equation?
Stephen:	y equals or ... something like that.
Interviewer:	To have it equals something?
Stephen:	Mm.
Interviewer:	Alright, last one. What about ... x equals 4.
Stephen:	That's an equation.
Interviewer:	Alright, is it a function?
Stephen:	uh ... no.
Interviewer:	Okay, so ... how would ... could you make it into a function?
Stephen:	Yeah, just ... x minus four equals something ... a y or zero.

Stephen's thinking about functions does not include the formal uniqueness property. It corresponds instead to a general notion of a rule or relationship which must be presented in a specific format. Thus, "x = 4" is not a function to Stephen, not because it fails the vertical line test, but because it is not written the right way.

Stephen's thinking about functions became more clear during the course of the project. Soon after the initial interview, he was able to state that "a function is a statement or rule [in] which you can use for any numbers to find an answer which comes out from using the function. [For example] f(x) = x^2 - 4x + 3, f(x) = 3x - 11". Much later, towards the end of the data gathering phase of the study, his thinking had clarified even further:

> A function is a way of describing a certain pattern. When substituting various numbers into it, it will give different numbers which have been changed according to the value of the function. f(x) = x^2 - 2, f(x) = 3x - x^2.

Stephen's early disjoint conception had solidified to a clear and versatile understanding of both nature and purpose. Later study of the images used to think about these concepts, however, revealed that, even at this stage, inconsistencies still existed within Stephen's understanding of these central concepts.

Of the two junior secondary students, after an introduction to the ideas of function, Tony remained unable to articulate a verbal definition - "I can do the questions but I don't know what it is," while Patrick demonstrated a more coherent grasp of the concept. Asked to try to give two different explanations of what a function is, he offered that "a function is starting off with a difficulty in the sum and then working out what the characters in the sum would need to do to solve it" (probably identifying function with equation), but then noted that "a function is being in a way related to someone or something," demonstrating a fundamental grasp of the central idea.

Summary

The various definitions of algebra, then, suggest a coherent and consistent view which appears to permeate all levels studied. An early understanding of algebra is most likely to be static, fixated upon its representative nature, defined by the use of letters replacing numbers.

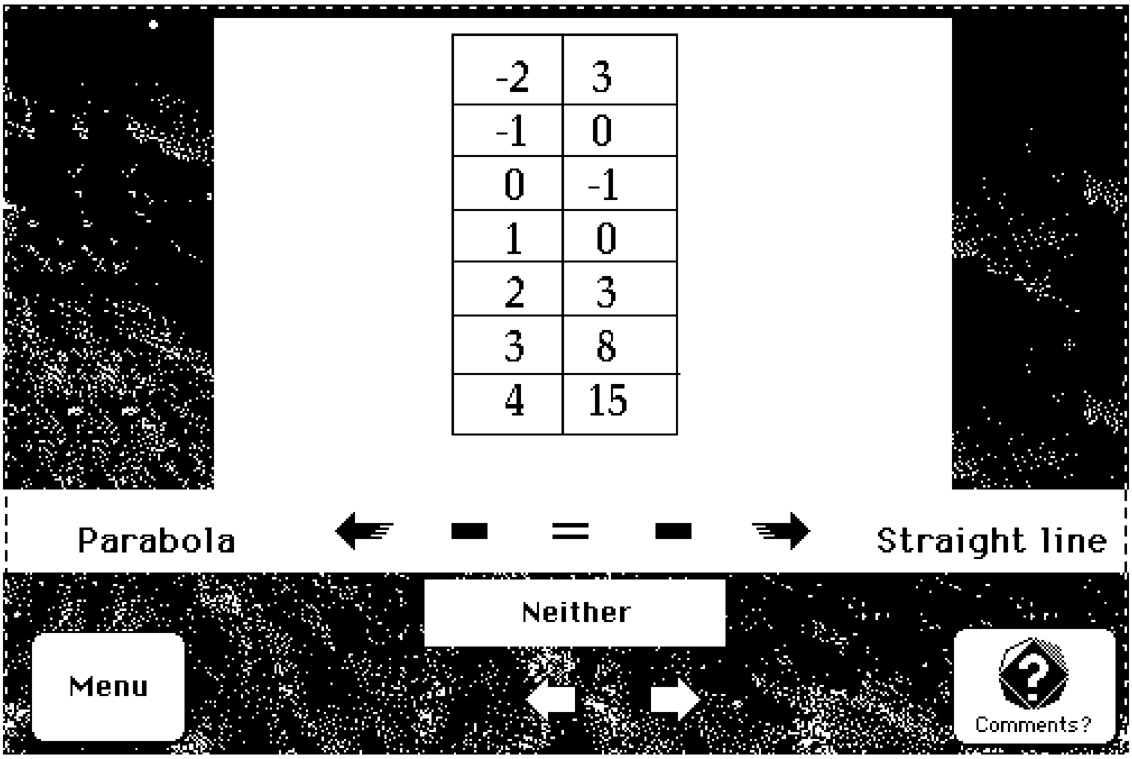
Later views of algebra are likely to incorporate a more active element as the previously representative object becomes a process of solving and finding answers. The use of additional representations, even among older students, was not common; of those mentioned, only the graphical form was offered as an alternative and, for some students, this did figure quite strongly in their thinking about algebra. Finally, algebra is seen to have no function beyond its own borders: its place in school and in mathematics is justified only by its own nature: algebra is studied in order to study more algebra. Such a view offers little motivation for its study for those for whom schooling is insufficient as an end in itself.

Understanding of algebra and its related concepts as demonstrated by verbal definitions appears to be quite fragile among all secondary students (and even some of the tertiary students), largely composed of disparate and poorly connected concepts. The need to expose the links between these ideas and the visual nature of many responses to algebraic concepts suggest that a study of algebraic imagery may reveal much more concerning the cognitive structures related to thinking about algebra.

Images of Algebra

What are termed here "images of algebra" were elicited through presentation of a series of ten cards, displaying a range of common algebraic visual prompts which participants responded to in a variety of ways. Participants were first asked to verbally describe each card, and then to sort them into as many groupings as they could (this may be considered a **first order grouping**). Four of the six Group A preservice teachers engaged in this process, as did all the secondary students. The secondary students then engaged in a more detailed discriminatory exercise, in which the ten images were presented three at a time, and students were asked to "choose the odd one out" - to decide in what way one image was different to the other two. This **second order grouping** exercise forced students to compare and contrast properties of the different images, and so potentially engage in a deeper analysis than the previous sorts.

Figure 6.2: Sample of a *RepGrid* analysis



As a final, in-depth analytical mechanism, Stephen, Tony, Patrick and the researcher (SMA/T1) engaged in a **third order grouping**, a detailed Repertory Grid analysis of the categories which arose from the previous discriminatory activity. Categories identified from the second order grouping were taken in pairs, placed as the end points of a continuum, and then presented with each of the ten original images. For example, I had distinguished between “parabola” and “straight line” in my second-order grouping. I was then asked to decide the extent to which each of the ten card images displayed these two properties by clicking at points between them (Figure 6.2). This process attempts to explicitly expose the network of relationships perceived by each individual in their thinking about algebra.

Images were chosen so as to offer the basis for sorts based both upon surface properties (for example, symbol/graph/numbers) and a range of possible other categories, such as functions/non-functions, different representations of the same symbolic form (cards 7 and 8) and even potential errors, such as equating the graph in card 2 with the visually similar symbolic forms of cards 7 and 8.

Before detailing the responses for each grouping, the initial verbal descriptions for each image are examined. These proved revealing of the level and nature of individual thinking regarding algebraic concepts, especially in comparison with the verbal definitions already described.

Card 1(expression): $4 - 3x$

It has been suggested that one likely impact of computer technology upon the teaching and learning of algebra will be to move the central object of focus for the algebra curriculum from equation to function (Fey and Good, 1985, Kaput, 1992). Certainly the majority of computer tools used in algebra learning take the function as their principal object of action. An expression such as **4 - 3x** may be viewed in several different ways which are relevant in this context. In fact, different individuals were found to read a surprising variety of signals into such a simple object.

Student teacher A2, for example, observed first that this was simply "not an equation as such" (suggesting the dominance of the equation as an algebraic object of focus). Later, she expanded upon this:

> This is a number statement that could show as the value of x increases, the answer will be getting smaller. As x decreases (gets closer to zero) the answer will be getting smaller but will not be less than zero. If the x value goes below zero then the answer will be getting larger.

Note both the level of generality of the response (unrelated to the actual visual features, such as the 4 and -3x components) and the active process orientation, in which the value of the variable "x" is dynamically linked to the "answer". A2 does not say that as the value of x changes, the answer "gets" smaller; her implication in using "getting" seems to be of a highly fluid, dynamic relationship. In SOLO terms, this response might be classified as indicative of *formal operational thinking*, comparable to van Hiele's *Theoretical level*.

The response of A3 was more typical of multistructural concrete-symbolic thinking associated with the algebraic object: "From the numeral '4', a value calculated as 3 times another number 'x', is subtracted. Any number can be substituted for x, and thus the equation has multiple values". Once again, the response carries with it an active perception of process, this time related to the sequence of operations implied by the algebraic expression and also associated with the "function machine" image, of numbers "going in" and other numbers "coming out". Note the assumption by both individuals that the unspecified expression can act as an "implied equation".

For preservice teacher, A4, the immediate response ("This doesn't mean jack shit to me") was replaced by a more reasoned one: "This is a function; when x is greater than 1 1/3 our answer is negative, when x is 1 1/3 our answer is 0, when x is less than 1 1/3 our answer is positive". He then goes on to comment: "I would feel more comfortable if the card read 4 - 3x =". Like A2, this individual looks beyond the visual properties to respond almost automatically using equation-solving techniques. The equation signal is so strong that the expression is viewed as incomplete without an equals sign.

Both A5 and A6 responded in the same manner as did A3, reacting to the surface stimuli of the expression and responding to the operational process implied: "To me this is a statement which involves an unknown quantity. It represents three lots of something being subtracted from 4" (A5) and "This means 4 minus three times the value of an unknown number. The x can take any value unless some restriction has been placed upon it. In this form the expression can not be simplified. It is simply a generalisation of the idea expressed in my first sentence".

The responses of the preservice teachers fall clearly into two groups - those who responded to the immediate visual signal of the object, and those who operated at a higher level of generality. In all cases, however, there appeared to be a level of confusion regarding the signal nature of the expression: all wanted to *do* something to it - to substitute, to simplify or, in the majority of cases, to solve. These are the responses provided by traditional training in algebra - the desire for closure may be satisfied by action upon the algebraic object, and only three strategies are available, even to these highly trained students of mathematics.

The responses of the students to the expression **4 - 3x** were similarly revealing. While Patrick admitted readily that "this doesn't really mean much to me because I haven't worked with any of this stuff yet," Tony observed that it was, to him, "an equation and undoable question". Notice once again the desire to act stimulated by the algebraic expression, a desire frustrated by a lack of available strategies by which it may be operated upon. Jane and Ben both reacted to the visual stimulus - "taking 3x away from 4" (Jane) and "a simple subtraction equation where x is unknown" (Ben). The more experienced students, however, vented their desire to act by relating the expression to the coordinate plane: "line, y intercept at 4, gradient 3" (Andrea) and "this equation makes me think of a straight line with a gradient of 3 and crosses the x-axis at 1 1/3" (Stephen). The readiness of these students to assume a graphical metaphor by which to conceptualise the expression appears likely to result directly from their increased exposure to the graphical representation within their technology rich learning environment. Notice that, for Stephen at least, this graphical metaphor actually subsumed and included the equation-solving view of the preservice teachers (since he had used these techniques to arrive at the value of the x-intercept).

I had seen it as all these things: "An expression which describes a numerical process, acting upon an infinite array of numbers to produce a new infinite array. Graphically, this represents a straight line with negative slope (of -3) passing through 4 on the y axis". This relational response links the various elements of the previous multistructural responses.

Overall, then, responses appeared to fall into three levels of increasing complexity:

(1) an immediate, visual response, reacting to the *symbol* nature of the expression;

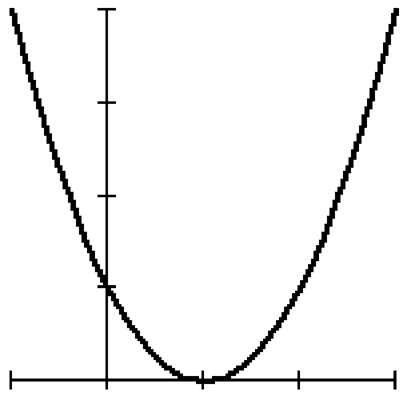
(2) a first-level signal response which views the expression as an equation to be acted upon, and

(3) a second-level signal response in which the object is viewed in both a cross-representational sense and as an "implied equation".

Card 2 (equation - *cf.* Card 9): $y = 2x - 1$

Participants were far more comfortable with this image than with the previous one. In every case (other than Tony - who saw it as an equation to be solved - and Patrick, who had insufficient algebraic experience to recognise it), the equation triggered the same response: as representing a graph with gradient 2 and y-intercept -1. The signal character of the equation in this form was strong and immediate.

Card 3 (parabola graph):



(Note that the symbolic form for this graph is $\mathbf{y = (x - 1)^2}$. This may be contrasted with both Cards 7 and 8, which offer different representations of a similar but distinct symbolic form.)

As with the equation, $\mathbf{y = 2x - 1}$, the responses were uniform and predictable. All recognised the signal of the parabolic form.

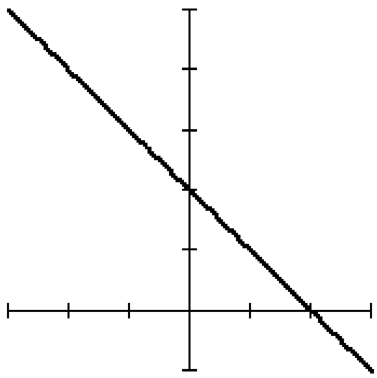
Card 4 (x, y pair): (x, y)

In all cases, this symbol was recognised as representing a point or position on the number plane. The signal was clear and unambiguous.

Card 5 (function symbol): $f(x)$

More varied were the responses to this symbol, ranging from the unistructural *"This is function x"* (A4) and *"This represents a function with respect to x"* (A5) to identification with numerical values (A3, A6, Andrea and Stephen), pairs of numbers (Ben) and points on a graph (Jane). The symbol appeared to trigger either an active numerical image or a more static graphical one.

Card 6 (graph of y = 2 - x)



Once again, a uniform response from all participants. All noted that it was a straight line on the number plane, and most identified gradient as -1 and y intercept as 2.

Card 7 (Table: $y = x^2 - 1$)

(Note that the factored form of this function is $(x - 1)(x + 1)$: *cf*. Card 8 and contrast with Card 3).

-2	3
-1	0
0	-1
1	0
2	3
3	8
4	15

Individuals responded in three distinct ways: focusing upon the table as points for a graph (Jane, Tony and Patrick), as input/output pairs (Stephen) or as the result of a rule or equation (A5, A6, Andrea and Ben).

Card 8 (expansion)

(*cf*. Card 7, contrast with Card 3)

$$(x - 1)(x + 1)$$

Responses to this card provide examples of what van Hiele refers to as a **rigid structure** - the signal to act by expanding to give the familiar "difference of two squares" was almost overpowering. All responded algebraically, while only Andrea, Ben and Stephen reacted graphically as well. Once again, the student teachers appear to react differently to the students who have had exposure to computer tools and, as a result, have become comfortable with the graphical representation.

Card 9 (equation)

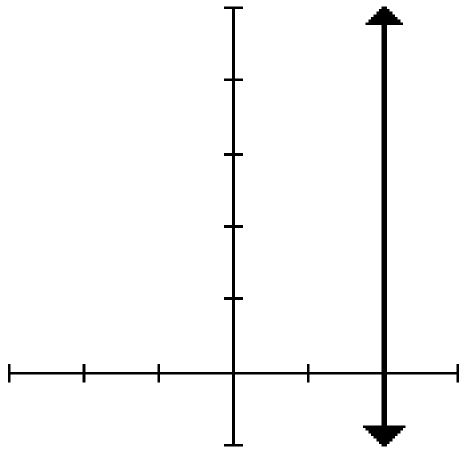
(*cf*, Card 2)

$$2x - 1 = x + 7$$

Like the expansion of card 8, participants appeared to react positively to this prompt, apparently feeling themselves to be once more upon familiar ground. All responded to the manipulative signal to act which had been so effectively instilled in them by their algebraic instruction. Interestingly, Andrea, Ben, Jane and Stephen all indicated solutions involving the physical movement of parts of the equation "across the equals sign".

Card 10 (graph of x = 2)

(Note that this is an example of a non-function). Participants saw it as a straight line, with most identifying it as x = 2. It triggered no other responses.



Analysis of the ten "images of algebra" across all participants and all grouping activities led to the identification of eleven principal **descriptors**, with several sub-categories related to the most common occurrences of some of these. These are displayed along with the participants from whom they were derived in Table 6.2.

Since all these participants responded to the same ten visual prompts to produce the descriptors listed, the table gives some indication of both the relative complexity of the thinking about algebra displayed by different individuals (viewing vertically) and the relative frequency of occurrence of the different descriptors (viewing horizontally).

Table 6.2

Algebra Descriptors

	A2	A3	A4	A5	A6	S1	S2	S3	S4	S5	S6	T1
1. Equation	√	√	√	√	√	√	√	√	√	√	√	√
1. linear	√		√	√	√	√	√	√	√	√		√
2. quadratic				√	√	√	√	√	√			√
3. Non-equation	√					√	√					√
2. Graph	√	√	√	√	√	√	√	√	√	√	√	√
1. linear	√	√		√	√	√	√	√	√	√		√
2. quadratic	√	√	√	√	√	√	√	√	√	√		√
3. non-graph						√	√		√	√	√	
3. Function	√	√	√	√	√	√	√	√	√			√
2. quadratic									√			
3. non-function							√		√			
4. Expression/symbol		√		√	√	√	√	√	√	√	√	√
5. Signal to action	√	√	√	√	√	√	√	√	√	√		√
6. Numerical value	√	√	√		√	√	√	√	√	√	√	√
7. Function machine	√				√		√	√	√			√
8. Relationship	√	√			√							√
9. Table of values	√	√		√	√		√	√	√	√		√
10. Variable	√	√	√	√	√		√		√	√		√
11. (x,y) Coordinates	√	√	√	√	√	√	√	√	√	√	√	√
- Non-coordinates						√			√	√	√	√

With regard to the individual participants, it is not surprising to find S6 (Patrick) displaying the most limited range of descriptors, along with S5 (Tony) and S3 (Jane). This offers validation to the findings based upon these student's verbal definitions, to suggest that their network of algebraic concepts is less well developed in comparison with the older participants. Even among the preservice teachers, differences in complexity are apparent, with A4 displaying the most limited repertoire of ways of thinking about algebra. Neither is it surprising to find the most extensive range of descriptors associated with myself as teacher/researcher, both because of my greater mathematical experience, and also because of my role in designing the research questions and analysing the results.

Most frequent descriptors across both students and preservice teachers were the common algebraic forms - equations, graphs and (x, y) coordinates. Distinctions between graphs - straight lines and parabolas - were also evident for all but Patrick (S6), and there was general recognition of functions, variables and expressions, although these terms were not always used appropriately (*equation* and *expression*, for example, tended to be interchangeable terms). Common, too, was an active conception of algebraic forms - the signal characteristic discussed previously. The identification of participant categories, however, was only a first step in the research design. The final aim was to make explicit relationships which might exist between these categories, and to expose the nature of the cognitive network by which the various individuals conceptualise algebra.

First Order Groupings

A useful measure of the complexity of individual thinking about algebra was offered by the first order groupings of the image cards. This process had an immediacy which tended to be revealing of the *signal nature* of the various algebraic forms. Although at times the groupings were idiosyncratic, there were clear patterns of consistency which went beyond the “surface” characteristics of the cards.

My own groupings, both as an experienced mathematics educator and as research designer, were, not surprisingly, the most comprehensive.

Six groupings were identified which provide perhaps as much insight into my perceived priorities within the research design as into my algebraic thinking.

Table 6.3

First order groupings: SMA/T1

Group 1: *Expressions*	**Group 2:** *Equations*	**Group 3:** *General forms*	**Group 4:** *Linear graphs*	**Group 5:** *Parabolas*	**Group 6:** *Pairs of numbers*	**Group 7:** *Non-functions*
4 - 3x (x-1)(x+1) *f*(x)	2x-1= x+7 y = 2x - 1	(x, y) *f*(x)	Gr (y=2-x) Gr (x=2) y = 2x-1 2x-1=x+7 4 - 3x	Table Gr (parab.) (x-1)(x+1)	Table (x, y) Gr (y=2-x) Gr (parab) y = 2x-1 Gr (x=2)	Gr (x=2)

The explicit recognition of non-functions, the association of graphs, table and equations as representing "pairs of points" and the inclusion of the equation **2x-1 = x+7** and the expression **4 - 3x** with other "linear graphs" signify a perspective quite distinct from that of the other participants. This difference will influence both the research design and the analysis which follows.

Of the preservice teachers, A2 produced only two groups from her single sort. These groups were mutually exclusive - all cards were included once except the *(x, y) pair* which was apparently overlooked.

Table 6.4

First order groupings: A2

Group 1: *Showing y as a function of x*	**Group 2:** *shows values of x*
y = 2x-1 *f*(x) Table of values Graph y = 2-x Graph parabola Graph x = 2	4 - 3x (x - 1) (x + 1) 2x - 1 = x + 7

A4 proposed four groups, reusing several prompts in the process.

Table 6.5

First order groupings: A4

Group 1: *Graphs*	**Group 2:** *Functions of y*	**Group 3:** *Coordinate Geometry*	**Group 4:** *Solving equations*
Graph (y = 2- x) Graph (parabola) Table of values Graph (x = 2) y = 2x - 1	(x, y) Graph (parabola) Graph (y = 2 - x) Graph (x = 2) Table of values	(x - 1)(x + 1) y = 2x - 1 2x - 1 = x + 7	4 - 3x (x - 1)(x + 1)

Group 1 suggests that, like Stephen, A4's thinking about algebraic objects is strongly influenced by their visual format. While the function symbol, the linear equation and even the graphs and table of values were seen as explicitly denoting a functional form, the two expressions and the equation to be solved were not. Rather, these were seen as representing values of x (and, by implication, *not* values of y). An expression such as **4 - 3x** appears to lack the signal character of an equation such as **y = 2x - 1**. Note, too, the incorrect inclusion of the graph x = 2 as a function.

It should be noted here that Groups 1 and 2 are essentially the same, with only the equation **y = 2x - 1** and the ordered pair **(x, y)** to distinguish them (even these appeared arbitrary, since the first could certainly be considered a "function of y" and the ordered pair is commonly associated with "graphs", consistent with A4's association of the table of values with the cards indicating graphs.) Again, the graph of the deliberate non-function, **x = 2**, was grouped with "functions", and the two **expressions** were incorrectly labelled as **equations** (Group 4). This last suggests that A4 may see an implied equation form, such as **4 - 3x = 0** when viewing the expressions. This grouping excluded the function symbol, ***f*(x)** and, overall, tended to suggest a poorly organised cognitive network related to algebra.

A5 and A6 showed surprising consistency in their cognitive organisation. Both created five categories, which included straight lines, parabolas, equations and graphs. Both recognised the **table of values** and the expression **(x - 1)(x + 1)** as representations of quadratic forms, suggesting an ability to move across symbolic, graphical and numerical representations, and indicative of a relatively high level of functioning in this domain. At the same time, neither clearly distinguished expressions from equations, nor explicitly differentiated functions from non-functions, even though all preservice teachers had included the uniqueness property within their verbal definitions of function. In fact, A2, A4 and A6 incorrectly placed the graph **x = 2** within groupings designated as "functions". Knowledge of the formal definition does not necessarily imply the ability to correctly apply it, a result which echoes the findings of Vinner and Dreyfuss (1989).

Table 6.6

First order groupings: A5

Group 1: *Straight lines*	**Group 2:** *Parabolas*	**Group 3:** *Equations*	**Group 4:** *Statements*	**Group 5:** *Graphs*
Graph (y= 2-x) y = 2x - 1 Graph (x = 2)	Graph (parab.) Table of values (x-1)(x+1)	y = 2x - 1 Graph (parab.) 2x-1 = x +7 Graph (x=2) Graph (y=2-x) (x-1)(x+1)	4 - 3x y = 2x - 1 2x - 1 = x + 7 (x - 1)(x + 1)	Graph (parab.) Graph (y = 2-x) Graph (x = 2)

Table 6.7

First order groupings: A6

Group 1: *Linear equations*	**Group 2:** *Quadratic equations*	**Group 3:** *Functions*	**Group 4:** *Equations*	**Group 5:** *Coordinates/ Graphs*
4 - 3x Graph (y= 2-x) Graph (x = 2) y = 2x - 1	Table of values Graph (parab.) (x-1)(x+1)	Table of values *f*(x) Graph (parab.) Graph (x=2) Graph (y=2-x)	y = 2x - 1 2x - 1 = x + 7	(x, y) Graph (y = 2-x) Graph (parab.) Table of values Graph (x = 2)

The groupings produced by the secondary students were similarly diverse. Andrea (S1) identified three categories only, but these spanned the symbolic and graphical representations. Thus she correctly grouped both graphical forms of parabola and straight lines with their symbolic forms, and included the expression **4 - 3x** and the equation **2x-1 = x+7** as straight lines. She was not, however, able to interpret the table of values in order to recognise its quadratic nature, nor was she able to place the ***f*(x)** symbol in relation to the other cards. For a student attempting the high level Three Unit course in mathematics, this grouping suggests a limited cognitive repertoire regarding algebraic understanding, but one within which the graphical representation plays a significant part.

Table 6.8

First order groupings: S1 (Andrea)

Group 1: *x & y coordinates*	**Group 2:** *Parabolas*	**Group 3:** *Straight lines*
(x, y) Table of values	Graph (parabola) (x - 1)(x + 1)	Graph (y=2-x) 4 - 3x y = 2x-1 Graph (x=2) 2x-1 = x+7

Like Andrea, Ben's sort appeared to be strongly influenced by surface characteristics of the algebraic images presented, although Andrea's categories related to algebraic properties, while Ben's arose from visual cues. There was little evidence of insight into connections between the various cards beyond that suggested by their visual appearance. As with most other participants, equations and expressions are considered interchangeable terms and, in Ben's case, the **f(x)** symbol is associated with graphs.

Table 6.9

First order groupings: S2 (Ben)

Group 1: *Expressions of algebra*	**Group 2:** *Graphs*	**Group 3:** *Pairs*	**Group 4:** *Equations which can be solved*
(x-1)(x+1) 4 - 3x 2x-1 = x+7 y = 2x-1	Graph (y = 2-x) Graph (parabola) Graph (x = 2) f(x)	Table of values (x, y)	2x - 1 = x + 7 y = 2x-1 (x-1)(x+1)

Jane engaged twice in the algebra card sort, offering four groups initially and seven groups later in the research process. She chose to use both graph plotter and table of values during the second sort to examine two of the images more closely (the graphical image of the line y = 2 - x, for which Jane used the graph plotter to ascertain the values of the intercepts, and the table of values card - after entering the values into the table of values utility, she then plotted these using the graph plotter). This use of tools supported increasing breadth and depth of analysis as suggested by the Vygotskian **Zone of Proximal Development**. Jane used the available tools to go beyond her current cognitive state and consequently recognised properties of the algebraic images which, while meaningful, were beyond that which she could have done unaided (she recognised, for example, the quadratic nature of the table of values image).

Table 6.10

First order groupings: S3 (Jane): Sort 1

Group 1: *Algebra*	**Group 2:** *Graphs*	**Group 3:** *Functions*	**Group 4:** *Equations*
4 - 3x (x, y) (x-1)(x+1) 2x-1 = x+7 y = 2x-1 *f*(x)	Graph (y = 2-x) Graph (parabola) Graph (x = 2) Table of values	(x, y) *f*(x) y = 2x-1 Table of values	2x - 1 = x + 7 y = 2x-1

Jane's first sort is surprisingly similar to that of Ben, both offering a "grab-bag" of algebraic forms in Group 1 (possibly anything with an "x" in it), and very similar groups for **graphs** and **equations**. Jane's use of the term **functions** as a category is interesting in the light of her uncertainty regarding its meaning which had been exposed in the earlier interview. This had perhaps sensitised her to the term and led to an increased awareness of its occurrence. Her choices for this group were all appropriate, suggesting that she is able to recognise examples of its occurrence to at least a limited extent.

Jane's second sort suggests increased cross-representational facility - the table of values is recognised as representative of both graph and parabola and the symbolic forms, **y = 2x - 1, (x, y)** and ***f*(x)** are readily associated with graphs as well as input/output numbers. It would appear that these algebraic images have moved through the use of software tools and from exposure to a technology-rich learning environment from possessing a **symbol nature** as demonstrated in sort 1 to acting as meaningful cognitive **signals** in the later sort.

Table 6.11

First order groupings: S3 (Jane): Sort 2

Group 1: *Graphs*	**Group 2:** *Algebra*	**Group 3:** *Equations*	**Group 4:** *Involve numbers*	**Group 5:** *Substitution*	**Group 6:** *Straight line graphs*	**Group 7:** *Points on a graph*	**Group 8:** *Parabolas*
Gr (y=2-x) Gr (par.) Gr (x=2) (x, y) Table f(x) y = 2x-1	(x-1)(x+1) 2x-1=x+7 y=2x-1 4 - 3x f(x)	2x-1 = x+7 y = 2x-1	4 - 3x Table (x-1)(x+1) y = 2x-1	f(x) (x, y)	Gr (y=2-x) Gr (x=2) y = 2x-1	Table (x, y) f(x)	Table (x-1)(x+1) Gr (par.)

Like Jane, Stephen (S4) also engaged twice in the algebra card sort activity and, also like Jane, increased over the intervening period from four categories to seven categories. He chose, however, not to make use of available software tools. His first sort was restricted in that he used each card only once, and so sorted them into exclusive categories. He displayed limited cross-representational facility, recognising the equation **y = 2x-1** as a linear graph and the expression **(x-1)(x+1)** as representing a parabola. He also treated the expression, **4 - 3x**, as an "implied equation", capable of solution if "**= 0**" is assumed as a suffix. Functions were included only as symbolic and numerical forms (the table of values implying for Stephen an "input/output" image suggestive of function).

Table 6.12

First order groupings: S4 (Stephen): Sort 1

Group 1: *Functions*	**Group 2:** Solving equations	**Group 3:** *Straight lines*	**Group 4:** *Parabolas*
f(x) Table of values (x, y)	4 - 3x 2x-1 = x+7	y = 2x-1 Graph (y=2-x) Graph (x=2)	(x-1)(x+1) Graph (parabola)

Table 6.13

First order groupings: S4 (Stephen): Sort 2

Group 1: *Function*	**Group 2:** *Straight line*	**Group 3:** *Parabola*	**Group 4:** *Equation*	**Group 5:** *Coordinates*	**Group 6:** *Equation for axis*	Group 7: *Find values for variables*
Table	Gr (y=2-x)	Gr (parab.)	4 - 3x	(x, y)	Gr (x=2)	4 - 3x
f(x)	Gr (x=2)	(x-1)(x+1)	2x-1 = x+7	Table	y = 2x-1	(x-1)(x+1)
	y = 2x-1		y = 2x-1		Gr (parab.)	2x-1 = x+7
			(x-1)(x+1)		Gr (y=2-x)	

Although Stephen demonstrated the most versatile and deep understanding of algebraic concepts of the student group, his second sort showed less improvement than did Jane's. His increased number of groupings appeared generally consistent but somewhat arbitrary (as in "Equation for axis"), and overall this sort demonstrated little improvement in his cognitive organisation than that which was made evident in the first. Although he showed good familiarity with the graphical representation, he appeared unable to interpret the table of values in a meaningful way.

Tony engaged in three first order sorts over a period of twelve months. Although the number of groupings increased in that period (from three to five), they remained based firmly upon superficial features of the images.

Table 6.14

First order groupings: S5 (Tony): Sort 1

Group 1: *Equations*	**Group 2:** *Number planes*	**Group 3:** *Things I don't understand*
(x - 1)(x + 1)	Graph (parabola)	4 - 3x
2x-1 = x+7	Table of values	*f*(x)
y = 2x-1	Graph (y = 2-x)	
	(x, y)	
	Graph (x=2)	

Interestingly, Tony's second sort reduced the number of groups to two:

Table 6.15

First order groupings: S5 (Tony): Sort 2

Group 1: *Graphs*	**Group 2:** *Equations and algebra*
Graph (y = 2-x)	4 - 3x
Table of Values	(x - 1)(x + 1)
Graph (parabola)	y = 2x - 1
(x, y)	2x - 1 = x + 7
Graph (x = 2)	*f*(x)

This second sort demonstrated improvement in both the appropriate use of technical terms ("graphs" instead of "number planes") and a clear distinction between what, for Tony, are the two fundamental divisions within algebra: symbols and graphs.

Tony's third sort displayed a finer detail and a better grasp of the language of algebra ("coordinates" and correct use of the term "expression") but not necessarily a deeper understanding of the distinctions between the various images. He now has a name for those "things I don't understand" from sort 1, and the expression **4 - 3x** has acquired the signal property of "something to be solved" which leads to inclusion with the equations - although, once again, the other expression, **(x-1)(x+1)**, was omitted (it had been seen as an "equation" in the first sort). Tony had studied expansion of binomials at school by this time, and indicated that he "knew what to do with this one", implying that he saw **(x-1)(x+1)** as something to be expanded. It seems possible that the stronger "expansion" signal served to "swamp" the "equation" signal in this case. Note that he correctly places the symbol ***f*(x)** as an expression, showing recognition of this form.

Table 6.16

First order groupings: S5 (Tony): Sort 3

Group 1: *Graph*	**Group 2:** *Equations*	**Group 3:** *Coordinates*	**Group 4:** *Table*	**Group 5:** *Expression*
Gr (parabola) Graph (y=2-x) Graph (x=2) (x, y)	y = 2x-1 2x-1 = x+7 4 - 3x	(x, y) Table of values	Table of values	4 - 3x (x-1)(x+1) *f*(x)

Finally, as a novice to algebra, Patrick's sorts would be expected to be based upon superficial cues, since the images possess for him no underlying meaning. His groupings reflect this.

Table 6.17

First order groupings: S6 (Patrick): Sort 1

Group 1: *Graphical*	**Group 2:** *Numbers and letters*	**Group 3:** *Mix*	**Group 4:** *Loner*	**Group 5:** *Problems*
Gr (parabola) Gr (y=2-x) Gr (x=2)	4 - 3x (x, y) (x-1)(x+1) 2x-1=x+7 y = 2x-1 *f*(x) Table of values	4 - 3x Gr (y=2-x) Table of values (x, y) Gr (parabola) y = 2x-1 *f*(x) Gr (x=2) 2x-1 = x+7 (x-1)(x+1)	*f*(x)	(x, y) 4 - 3x y = 2x-1 2x - 1 = x + 7 (x-1)(x+1)

Even at this early stage, Patrick recognised that algebraic forms possess a signal nature: his last group, "Problems", indicates objects which he saw as requiring some action, although he was unsure of what that action should be. His second sort was much more clearly defined than the first, with three clear and distinct groupings which again reflect the surface features of the algebraic forms.

Table 6.18

First order groupings: S6 (Patrick): Sort 2

Group 1: *Graphs*	**Group 2:** *Equals*	**Group 3:** *Numerals*
Graph (y = 2-x)	(x-1)(x+!)	4 - 3x
Graph (parabola)	2x-1 = x+7	(x, y)
Graph (x=2)	y = 2x-1	Table of values
		f(x)

Patrick's inclusion of the expression **(x-1)(x+1)** within the group labelled "equals" and the presence of the symbolic forms **(x, y)** and ***f*(x)** (and even the expression **4 - 3x**) among "numerals" reveals his recognition of the implicit and symbolic nature of algebra, where the symbols possess meaning beyond their surface appearance. Such a recognition appears to signify an important step forward in algebraic understanding. This understanding is probed further through the use of deliberate comparative techniques which give rise to the next level of image sorts.

Second Order Groupings

The deliberate comparing and contrasting of algebraic images (picking the "odd one out") offers an added degree of depth to the analysis of participant responses, forcing them to go beyond the often-superficial viewing associated with a verbal description. In particular, respondents who had difficulty in supplying verbal descriptions are provided with a non-verbal means of conveying elements of their thinking about algebra. At the same time, these non-verbal responses are supplemented by comments regarding the choice made, which provide further insight into the reasons for these choices (text records for these comments are recorded in Appendix E). Each of the student participants (except Jane) provided responses to this activity (Jane had left the program before this instrument had been developed); my own responses (SMA) are included as a further bracketing device (Table 6.19)

Table 6.19

Second Order Groupings: *"Pick the odd one out"*

		Andrea	Ben	Stephen	Tony	Patrick	SMA
1	4-3x y = 2x-1 Gr (y=2-x)	**1**	**3**	**1**	**1**	**3**	**2**
2	Gr (par) (x-1)(x+1) Table	**1**	**1**	**3**	**2**	**1**	**1**
3	(x, y) 2x-1=x+7 Table	**2**	**3**	**2**	**2**	**3**	**2**
4	*f*(x) 4 - 3x Gr (x=2)	**2**	**2**	**2**	**2**	**3**	**3**
5	(x-1)(x+1) 2x-1=x+7 4 - 3x	**1**	**1**	**1**	**1**	**3**	**2**
6	Gr (y=2-x) Gr (par) Gr (x=2)	**2**	**2**	**2**	**2**	**3**	**3**
7	y = 2x-1 (x, y) 2x-1 = x+7	**2**	**2**	**3**	**2**	**2**	**1**
8	y = 2x-1 *f*(x) 2x-1 = x+7	**2**	**2**	**3**	**2**	**2**	**1**
9	Gr (y=2-x) Gr (par) Table	**1**	**2**	**3**	**1**	**3**	**1**
10	*f*(x) (x, y) (x-1)(x+1)	**3**	**1**	**3**	**3**	**3**	**3**

The sorting of the ten images into triads was, in most cases, deliberate rather than random. Triad 1 grouped the three major algebraic categories: expression, equation and graph. Ben and Patrick both used an immediate visual distinction, and I distinguished on the basis of negative gradient. The remaining students associated **y = 2x-1** with the graph and saw the expression as the "odd one out".

Triad 2 grouped graph, table of values and expression, with mixed responses. The direct correspondence between the rule for the table of values and the algebraic form was recognised only by myself and Andrea, demonstrating a high level of cross-representational facility on her part. She had learned to interpret the table as a representational form. Ben also nominated the graph as the "odd one out", but for the superficial reason that both table and expression had "x values of -1 and 1". Stephen recognised that both graph and table represented parabolas, while Tony equated the tabular form with the graphical form. This item exposed a distinct hierarchy of responses:

Level 0: Patrick and Ben, who responded at a purely visual level,

Level 1: Tony saw the number pairs of the table as a general signal to graph these points.

Level 2: Stephen transferred meaning across the graphical and tabular representations, but on a visual level only.

Level 3: Andrea transferred meaning on a symbolic level.

The third triad demonstrated that, for most participants, the link between the tabular representation and the ordered pair symbol, **(x, y)**, was a strong one. Although, once again, Ben and Patrick responded in the same way, they did so for different reasons. Patrick focused upon the superficial "equals" prompt, which he saw as implied in the ordered pair but not in the table of values. Ben, on the other hand, contrasted the "known x and y values" of the table with the unknowns of the other two forms.

Triad 4 exposed the strong link between the ***f*(x)** symbol and the graphical representation for all students but Patrick. Patrick distinguished the graph from the "non-graphs", while I distinguished functions from non-function.

Triad 5 was similarly consistent across the students, but for a range of reasons. Andrea and Ben responded to the graphical representation implied, distinguishing between quadratic and linear forms. Tony responded to the "expansion" signal of the expression, while I focused upon expressions as opposed to equations. Stephen's verbal response to this item sees him again view the expression **4-3x** as an "implied equation", while **(x-1)(x+1)** does not have this property: "For the two that are alike you find an x-value, and with the other one it just describes a function". Clearly, for Stephen, the expression **(x-1)(x+1)** possesses a much more limited signal character than the linear expression.

It is hardly surprising that item 6 produced such a consistent response - the visual signal of curved/straight, parabolic/linear produced a prompt reaction from all students but Patrick, who saw the vertical line as somehow different to the other two (but could not say why), while I focused upon the function/non-function distinction.

A similar strong visual signal was sent by the equals sign in both Triads 7 and 8. Stephen's response to both items, however, was strongly influenced by the graphical representation which he brought to the question, linking **y=2x-1** and **(x, y)** in Triad 7 ("The similar ones are describing a straight line with two pronumerals (x and y), and the other one just defines an x-value as a number not in terms of y"), and **y=2x-1** and the symbol ***f*(x)** in Triad 8 ("The similar ones are functions and describe a graph or what you get if you put x-values in and make a picture, and the odd one out just finds the x-values"). Once again, Stephen's strong reliance upon the function-machine metaphor for thinking about functions is evident here.

Item 9 caused a range of responses: Andrea and Tony both demonstrated again that they are able to interpret the table of values representation, distinguishing parabolas from straight lines; Stephen and Patrick both responded superficially upon the basis of graphs and table forms, while Ben chose the parabola as the odd one out since it does not go below the x-axis, whereas the other two do. This demonstrates a limited ability to interpret tabular information.

Ben was also the "odd one out" in his response to the last item, seeing ***f*(x)** as a more general symbolic form than **(x, y)**: "The other two could both be just points, while *f*(x) could be anything". The other respondents accorded both symbols equal levels of generality, as opposed to the more specific expression.

The overall impression gained from these second order groupings is that of students who are all able to demonstrate quite strong cross-representational facility, at least across symbolic and graphical forms. Interpretation of the tabular form is much more limited, consistently found only in Andrea and apparently all but absent in Stephen, who prefers to think graphically. Along with Tony and Patrick, Ben displays a very limited representational repertoire.

Third Order Groupings

The task which gave rise to the third order groupings was a very time-consuming one, taking up to thirty minutes to complete. For this reason, only four participants were engaged in this activity - myself, Stephen (as the principal informant) and the two junior secondary students, Tony and Patrick (whose limited algebraic experience meant that they had been restricted in their access to appropriate language and forms of expression by which their understanding might be examined). The previous tasks in these two cases had furnished limited information regarding their algebraic thinking - it was hoped that this detailed analysis might provide a useful non-verbal vehicle by which their cognitive frameworks might be better assessed.

This task involved three steps:

1. The verbal statements which had accompanied the second order grouping process were examined, and used to give rise to a number of descriptors which appeared to figure prominently in their thinking about the algebraic images. This process of extraction took place in collaboration with the informant, increasing validity for the descriptors.
2. The descriptors were entered into the *HyperCard* RepGrid stack and participants would again view each card individually. This time, however, instead of requiring a verbal descriptor, the descriptors would be displayed in pairs, as the ends of a continuum (see Figure 6.2).
3. Participants would choose an appropriate response which situated the given image in relation to the two descriptors.

In my case, the ten comments which accompanied my second order grouping process (Table 6.19) were:

1. They both have negative gradient.
2. They both represent $x^2 - 1$, while this one is $(x-1)^2$.
3. They imply an infinite set (or at least multiple ordered pairs); the equation implies a single pair.
4. This is a non-function; the others are functions.
5. The others are expressions (implying an infinite array of values); this is an equation.
6. Non-function.
7. The ordered pair representation for the others is explicit in x and y; for the equation it is implicit.
8. The other two each represent individual functions; the equation represents the equality of two distinct functions.
9. The others both represent parabolas.
10. Both the others are purely symbolic forms.

These comments gave rise to seven descriptors:

- function
- non-function
- equation
- expression
- graph
- table of values
- symbols

Stephen's comments for each of the ten triads were as follows:

1. The other two both have two pronumerals (x and y) but 4-3x has only one. It doesn't tell you what comes out if you put numbers in.
2. The other two are both parabolas and the other one is just a set of values which you put in and something comes out.
3. The other two both have one value going in and another coming out, but in the third one it is just finding the x-value.
4. The others are a function with special values - whatever you put in you get different answers, but in the third one it's just a statement - it doesn't show what you get out.
5. For the two that are alike you find an x-value, and with the other one it just describes a function.
6. The two alike are just straight lines, and the odd one out is a parabola.
7. The similar ones are describing a straight line with two pronumerals (x and y), and the other just defines an x-value as a number not in terms of y.

8. The similar ones are functions and describe a graph or what you get if you put x-values in and make a picture, and the odd one out just finds the x-values.
9. The other two are graphs on x and y axes, and the odd one out is a table of values - you don't really know what it is describing, which side is x or y.
10. The odd one out is not equal to anything, and the similar ones are describing a function.

From these were derived seven descriptors which encapsulated what for Stephen appeared to be key concepts regarding algebra:

- two pronumerals
- parabola
- function
- straight line
- graph
- table of values
- equation

Tony's comments were:

1. Because one had an x and a y in it, and the other was a graph - they both use x and y coordinates.
2. Because the table represents a graph and that ones a graph.
3. Because x and y are coordinates and the table is used in representing graphs.
4. Because the f(x) one has something to do with graphs .
5. Because the odd one out involves expansion and the other two are simple equations.
6. Because they are straight and the other one is a parabola.
7. Because the other 2 are equations and the one in the middle is a coordinate.
8. Because the other 2 are equations (again) and the odd one is a coordinate.
9. Because the table plots a parabola out and the other one is a straight line
10. The top 2 represent something to do with graphs and the bottom one has something to do with expansion.

These led to the identification of eight descriptors:

- graph
- coordinates
- table
- x and y
- equations

- expansion
- parabola
- straight line

Finally, Patrick's comments are given:

1. the other 2 are numerals but this one was a picture
2. I'm not sure about this one so I'm skipping this one.
3. The other 2 are describing how one thing equals another.
4. This one is a graph but the other 2 aren't.
5. I don't know.
6. the other 2 are marked by certain coordinations but this one isn't.
7. The other 2 describe what eg. x=, y=.
8. This is a symbol the other 2 aren't.
9. the 2 matchies are picture graphs but this one isn't.
10. the others are symbols but this one isn't.

Once again, seven descriptors were identified as arising from these:

- numerals
- picture
- not sure
- graph
- equals
- coordinations
- symbol

The full text of the third order groupings for each individual is recorded in Appendix E.

Analysis of the responses for each image card allows quite detailed assessment of individual thinking in relation to that concept. Responses for each card will be considered for the four respondents.

For myself, the algebraic expression **4 - 3x** elicited responses which classified it as symbol, function, expression, graph and table of values, but not an equation or non-function. It was seen as possessing characteristics of both graph and table of values, although more symbol than either of these. For Stephen, however, it was an image which was largely devoid of meaning - it was not associated with any of the categories chosen. It had no graphical or table of values form, was not a function (since it did not possess the "y=" prefix) and was not even associated with a straight line. Stephen's response to this item not only clearly highlights his erroneous conception of function, but demonstrates that the expression as an algebraic form has little signal character.

The expression elicited a similarly negative response from Tony, although he did associate it with "equation" and, interestingly, with table of values but not graph. For Tony, the table of values appeared to be perceived as a more flexible representation than the graphical form, more closely related to both formulas and numerical values.

For Patrick, the expression was dominated by its association with numerals, coordinations (Patrick's expression for the "x" and "y" coordinates of algebra) and its symbolic nature. It was not linked to graphs, pictures or "equals" (equation forms). He perceived the other expression, **(x-1)(x+1)**, in exactly the same way.

The symbolic **(x, y)** pair was perceived by me as representing equation, graph, table of values and symbols, but not function, non-function nor expression. The explicit "x" and "y" elements implied an association of equality which could be represented in various forms, but excluded the notion of expression which was considered to be defined by its *lack* of an explicit y-variable. Function and non-function links were rejected because the pair could equally represent either form.

For Stephen, the association of the ordered pair with two pronumerals was an obvious one, but it possessed partial characteristics of other responses. It was, for example, considered to be “more two pronumerals than straight line or equation”, and yet “more graph and table of values than two pronumerals”. It was seen to possess some elements of parabola, straight line and function, and equally both graph and table of values. This partial categorisation suggests a more mature and flexible cognitive network than those which are strictly “black and white”, in which each item is purely one thing or another (which was more evident among the two younger students). Stephen’s ability to perceive in algebraic images various “shades of gray” suggests that his thinking is relatively rich and diverse.

For Tony, the **(x, y)** pair was strictly “coordinates” and “x and y”, and possessed characteristics of both graph and table of values, but not of equation, expansion, straight line or parabola. For Patrick, the image was “coordinations”, “numeral” and “symbol”, with some lesser association with “graphs” and “equals”.

The responses to the three graphical images were quite consistent for all participants. In my own case, I distinguished between functions and non-functions, and noted that the images possessed an element of the table of values form. Tony, similarly, saw the graphs as “more graph than table of values”, while Stephen rated them equally - for Stephen, a graph was as much associated with the numerical values from which it was derived as with the graphical form. Patrick, too, recognised the role of "coordinations" as of equal importance in determining the graph as the visual form.

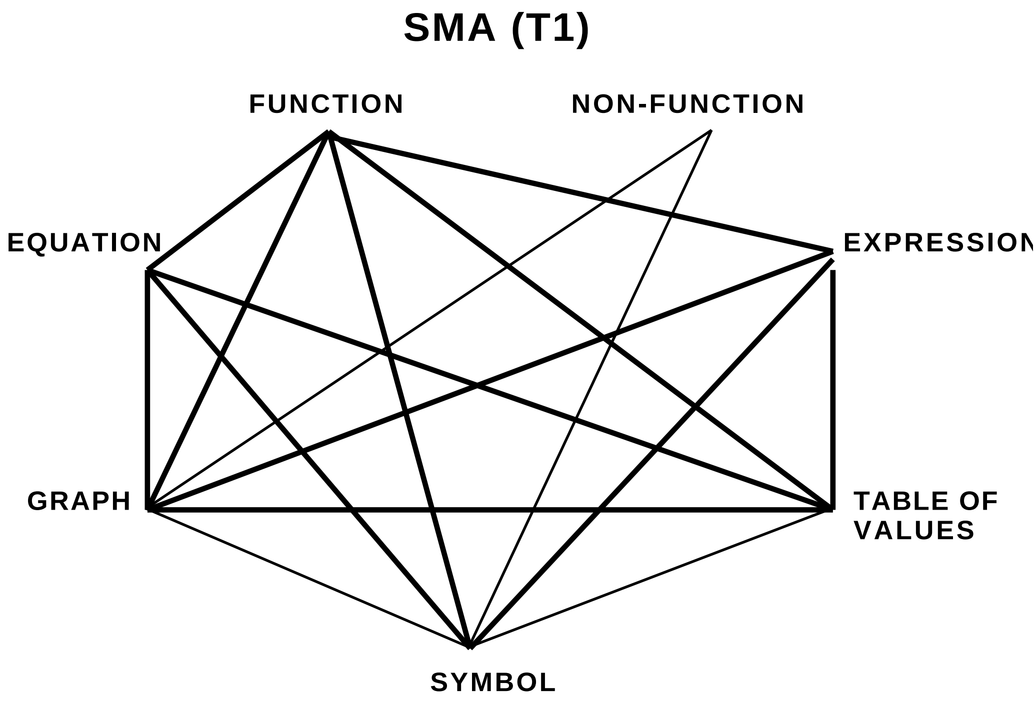
The ***f*(x)** symbol drew some varied responses. While I associated it with function, expression, graph, table and symbol, I rejected association with equation, since the required "=" symbol is absent. Stephen, on the other hand, saw it as linked to everything, possessing elements of equation, function, graph, table of values, even parabola and straight line. While Tony recognised a relationship with graphs, tables and coordinates, he rejected links with equation, straight line and parabola. The presence of the parentheses probably accounts for his perceived link with "expansion". Finally, Patrick saw it as a "symbol" with links to "numerals" but nothing else.

The equation **y = 2x - 1** was associated with graphs, tables of values and straight lines by myself, Stephen and Tony, while Patrick linked it with numerals and "equals". The other equation (**2x - 1 = x + 7)**, however, led to more diverse responses. I saw it as an equation with elements of graph, symbol and table of values. Tony and Patrick linked it only to "equation" (or "equals" in the latter case), with Patrick recognising a numerical component. Both rejected association with graphical and tabular forms.

Stephen's response to the equation was somewhat surprising since, after rejecting association with function, "two pronumerals", parabola, straight line and the graphical representation, he explicitly linked it with the table of values. Further, this was a clear and deliberate relationship, which he identified several times in the task. In particular, he explicitly rejected the graphical form but chose the tabular. Until this point, the two representations had appeared to be interchangeable in Stephen's way of thinking (although the table had appeared a little more flexible). Clearly this item demonstrates that, to Stephen, an equation *is* a table of values and that this form, unlike the graphical form, may not require a specific format for the algebraic object. Stephen's diverse understanding of the tabular representation was further highlighted in his responses to that particular image, in which he associated it with function, equation, graph, parabola and two pronumerals. Responses for Tony and Patrick regarding the table of values card were more limited, relating it to coordinates and graphs but not equations in both cases.

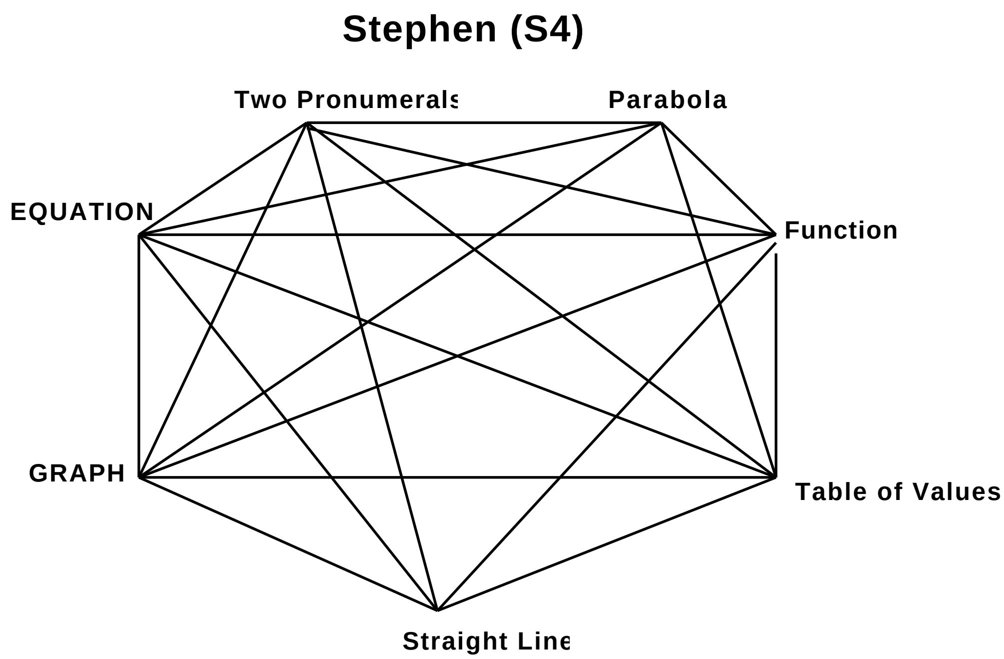
The networks of relationships derived from this task proved highly informative regarding the algebraic thinking of the various individuals involved. While the previous sorting tasks had allowed the identification of the various categories by which algebraic objects were conceptualised, this final task allowed these categories to be located within a dimensional space. Summary diagrams of these concept networks are presented in Figures 6.3 - 6.6.

Figure 6.3: Concept Network for SMA



The illustration of my own concept network depicts well-developed links for most descriptors. Some measure of this complexity is provided by the number of links associated with each node, or descriptor (note that the heavier lines indicate a strong connection across at least four different images; thinner lines indicate that the link was found in only one or two image cards). By this measure, the GRAPH and SYMBOL descriptors are the richest, associated with the widest range of algebraic concepts. The table of values is slightly less diverse, since the graph was found to be associated with a non-function, whereas the table of values representation was not. Similarly, EQUATIONS and EXPRESSIONS were strong but mutually exclusive categories, as were FUNCTIONS AND NON-FUNCTION. The network is suggestive of a relational understanding of algebra, in which the various components are perceived as meaningful, both in themselves and in relation to each other.

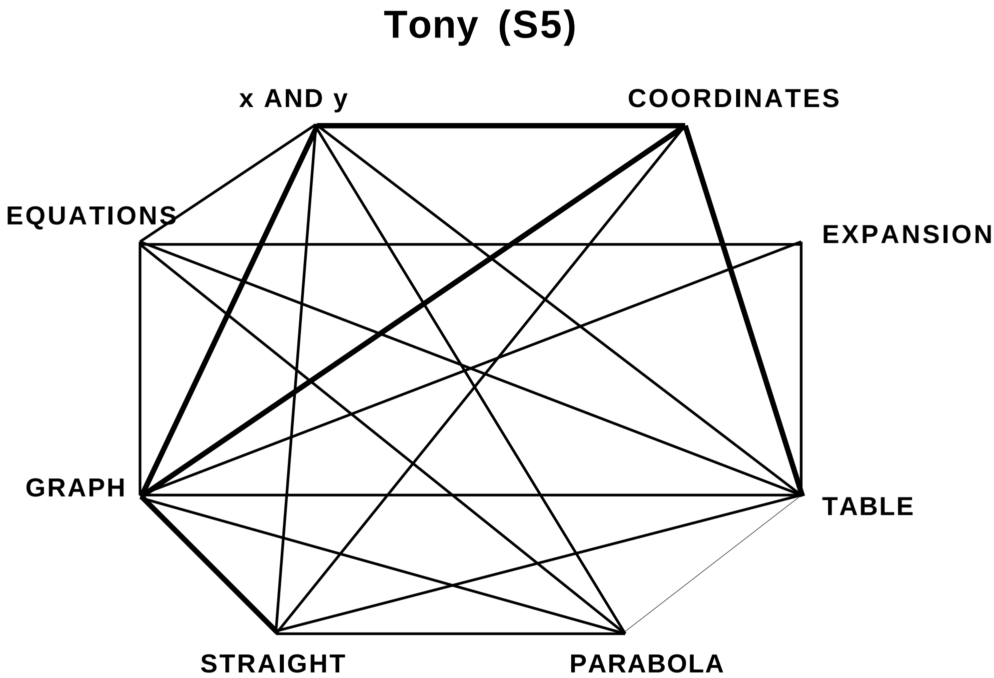
Figure 6.4: Concept Network for Stephen (S4)



Stephen's network of concepts displays well-developed links between all major categories, suggesting good algebraic understanding. His graphical thinking appears to be better developed than that associated with the table of values, with EQUATIONS and TWO PRONUMERALS being most extensively related. Stephen's thinking appears more "black and white" than that of the researcher - his concept links appear more of the "all or none" kind, suggesting that he distinguishes less clearly between them (as in his use of the terms "function" and "equation"). Although his understanding is best described as *relational*, it is clearly of a different order to that of myself.

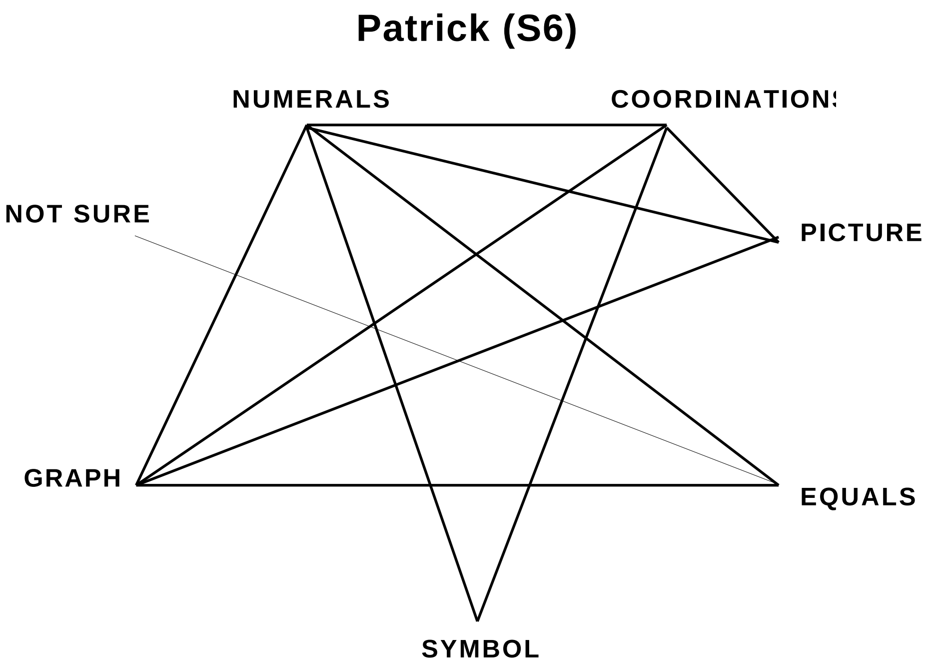
The differences between the concept network for Stephen and those for Tony and Patrick are immediately clear. While the younger students might have identified as many descriptors, these are poorly developed and associated constructs. Their relationships with other concepts is tenuous at best, illustrative of multistructural understanding at best.

Figure 6.5: Concept Network for Tony (S5)



It is hardly surprising to find Patrick's concept network to be even more limited than that of Tony, sure only that algebra involves pictures, graphs, numerals and symbols. Like Tony, repeated descriptors may be recognised: "x and y" and "coordinates" for Tony, "pictures" and "graphs" for Patrick. Patrick's few strong links are those between NUMERALS and SYMBOLS, and PICTURES and GRAPHS (demonstrating the symbol nature of algebraic forms). Clearly, for Tony, graphs are more meaningful objects than for Patrick, even to the recognition of the symbolic connections between equations and expressions and their graphical forms.

Figure 6.6: Concept Network for Patrick (S6)



The network diagrams shown above provide immediate visual clues as to the cognitive organisation of the participants. They illustrate both the nature and the relative strength of the relationships between the various constructs which go to make up each individual's cognitive network within the domain of algebra. Across all participants involved in the study, it is now possible to recognise important and detailed features of their algebraic thinking as a result of the intricate examination which has been described. The cognitive profiles which have been developed provide valuable insight into the algebraic thinking of these individuals which will give guidance in our further study of the role of computer tools in the algebra learning process.

An overview of algebraic thinking

It is useful at this point to draw together the extensive and varied findings related to algebraic thinking elicited so far. Although many aspects of such thinking are striking in their consistency across not only age and ability groupings, but even across student and preservice teacher groupings, other features define distinct individual conceptions of algebra for the various participants. It is possible at this stage to compare and contrast the algebraic thinking of the various groups of individuals who provided data for this study.

The preservice teachers displayed algebraic thinking which was generally versatile and reflected both traditional considerations - rules, formulas, variables - and some elements clearly drawn from their pedagogic studies, especially the role of patterns and concrete materials in algebra learning. Explicit recognition of formal aspects of algebra (especially the uniqueness property of functions, notions of domain and range and active conceptions of variable) suggest mature and well-connected understanding. At the same time, alternative representations appeared to play little part in their conceptions of algebra, with only A3 specifically mentioning graphs in relation to definitions of algebra, and tables of values in relation to functions. The grouping tasks indicated considerable variation in the depth of thinking inspired by the various images of algebra, from quite superficial connections on the part of A2 to relatively well-connected and rich groups displayed by both A5 and A6. Even at this level of mathematical achievement, there appears to exist quite significant diversity in the depth and connectedness of algebraic thinking among those who will soon be teaching the subject, with the majority clearly dominated by traditional perceptions and manipulative aspects of the study of algebra. As with the student participants, particular images were associated with strong and well-developed signal characteristics while others (especially the simple expression and table of values) appeared to be poorly connected in relation to both other representations and appropriate action strategies.

Of the student participants, Stephen and Andrea appeared most similar in the majority of their responses to both definition-based and image-based research activities. Each recognised algebra as serving both a representational role and an active "solving" role. Each showed good facility for relating symbolic and graphical information (correctly interpreting the algebraic expressions **4 - 3x** and **(x - 1)(x + 1)** in terms of their graphical features) and each explicitly recognised variables as representing multiple unknowns. In contrast, Ben saw algebra as fulfilling only a "solving" function, failing to recognise its usefulness in representing information in a variety of ways. Although Ben was functionally able to recognise and interpret the graphical representation, he appeared limited to this mode, unlike his peers who were able to operate across symbolic, graphical and numerical representations. Ben demonstrated a preference for graphical imagery when interpreting algebraic information, a reliance which proved useful for him on many occasions, but also hampered his ability to act and think flexibly when required.

Stephen displayed a preference for an active "function machine" image when thinking about and describing algebraic ideas. This image involved numbers "going in" to a symbolic expression and other numbers "coming out". Such thinking proved to be far more flexible than Ben's graphical image, readily supporting cross-representational thinking. In fact, the function-machine image appears to serve as a link across the three major representations, supporting versatile thinking which translates readily across numerical, graphical and symbolic forms. In Stephen's case, this powerful image was somewhat limited by his perceived need for a particular explicit notational form, particularly for functions (which required an "*f*(x) =" prefix) and graphs (requiring a "y =" prefix). Algebraic forms without such prefixes (such as **4 - 3x**) were emasculated for Stephen (and for others, including the preservice teachers), having nowhere to place the required "output" number, and so failing to support any sort of effective algebraic action. Interestingly, although Stephen indicated a willingness to see in such an expression an "implied equation" (solving it as if it read "4 - 3x = 0"), he was unable to concede that it could as easily possess an "implied y-value", allowing both graphical and tabular representations. This was *in spite of the fact* that both graph plotter and table of values utilities supported entry in this simpler expression form as well as the more usual "y =" form. It can only be assumed that Stephen's exposure to this flexible input feature was too limited to overcome his persistent belief that graphs and functions require particular algebraic formats. This belief may have been inadvertently reinforced by the instructional modules of the computer-based algebra learning environment, which automatically expressed algebraic objects to be graphed using the "y =" prefix, even when this was not present in the displayed form. Thus, clicking on an expression such as **4 - 3x** when encountered within the modules automatically led to the graphing of the equation **y = 4 - 3x**.

While Ben and Stephen displayed clear preferences for particular algebraic images, such thinking was not as clear for the other participants. Andrea demonstrated strong cross-representational facility, transferring meaning across all three representational modes, but she did not display the explicit "function-machine" imagery which distinguished Stephen's thinking. In fact, in the absence of an explicit preference for graphical thinking (as shown by Ben) or numerical thinking (as shown by Stephen), Andrea appeared comfortable with the symbolic form, from which she was readily able to deduce the other representations. Even the table of values (with its obvious link to the input-output form of the function-machine image for Stephen) was related by Andrea to the rule or equation which gave rise to it. Although logically one might expect that Stephen would display a preference for the table of values as a mathematical tool, based upon his preference for numerical imagery, he demonstrated limited use of this mode. In fact, its main function for Stephen appeared to lie in the display of the two "sides" of an equation, allowing solution by numerical methods. It is possible that his preference for an input-output metaphor for algebraic thinking may have left Stephen with "nowhere else to go" when confronted by a table of values, whereas for Andrea, such a form immediately suggested a symbolic rule or equation as its source.

Jane and the two younger student participants appeared to relate most strongly to the visual forms of algebra, particularly the symbolic notation by which it was most readily recognised. While Andrea saw beyond this symbolic form and related it to a relatively rich network of associations, this facility was all but absent among these younger respondents. For them, algebra was static - representational in nature only to the extent that it involved "letters standing for numbers". While graphs, ordered pairs and symbols were recognised as the components of algebra, the links between these elements were all but non-existent. Certain automated action processes were evident, but these were largely unsupported by understanding, and easily confused. The beginnings of links and relationships between the components of algebra was evident, but their development is clearly an extended process, even with the support of computer tools.

The study of algebraic thinking which has been described, then, suggests several important implications which may impact upon the use of available software tools. Individuals were shown to interpret different algebraic forms in a variety of ways. Certain forms displayed strong and consistent signal characters which readily led to action on the part of participants, both students and preservice teachers. Most notable of these were the two forms of equations encountered - **y = 2x -1** and **2x - 1 = x + 7**. The former was invariably associated with graphing, and the majority of participants demonstrated the ability to deduce useful graphical meanings (most particularly gradient and y-intercept information) from this algebraic form. The second induced in all participants an automatic action sequence, leading to the production of a “solution”. Both students and preservice teachers displayed some preference for solution by “physical” manipulation of terms, as opposed to the process of acting equally upon “both sides” of the equation.

While such forms displayed strong signal character, the simple algebraic expression **4 - 3x** induced frustration and confusion among even the most experienced participants. All expressed a desire to act in some way upon the expression, but were unable to do so with their existing repertoire of available mathematical actions. The expression **(x - 1)(x + 1)** was not associated with the same reactions, since it permitted a familiar mathematical action (expansion), and so induced a sense of closure. Since the majority of algebra software tools take such expressions as their principal objects of action, the lack of a strong and consistent signal character associated with this common form appears likely to significantly influence the use of computer tools.

Finally, preferred algebraic imagery was found to be of significance for several participants. While such images as the "function machine" were found to be flexible and to support cross-representational thinking far more effectively than a graphical image, it appears that over-reliance upon any one form may limit the effectiveness of algebraic thinking in general, and the use of representational software tools in particular.

In order to build a more complete picture of the use of available software tools by the participants in this study, however, it is necessary to go beyond considerations of algebraic thinking alone. The beliefs and attitudes of individuals regarding the learning of algebra must play a significant role in determining the nature and frequency of such use, and these must now be examined.