# *Nine*
# *Toward a Grounded Theory*

The many pieces of the puzzle have now been laid out and examined. Their properties have been teased out, leaving only the final stage of the process - putting the pieces together in relation to each other in order to form a coherent whole. To this end, the grounded theory method offers what is termed the “paradigm model” as a guide to those features considered most important in developing a theory which is to be dense and coherent, offering an integrated and explanatory description of the phenomenon under consideration. Such a theory has been lacking up to this point, with concerns related to classroom use of technological tools taking precedence over the necessary study of individuals interacting with mathematical software.

The action research problem which gave rise to this study involved “learning to use new tools”. The grounded theory of software use proposed offers substantive contribution in this regard. For practitioners seeking to use such tools to enhance their own teaching and learning of algebra within technology-rich environments, the detailed case study descriptions and the subsequent grounded theory allow them to experience vicariously the interactions and encounters, the successes and failures of the teacher as researcher in this context. They may then judge for themselves the extent to which these experiences are congruent with and informative of their own situations. The study offers, too, a wealth of detail regarding a new and rich aspect of mathematical pedagogy. It suggests new questions and new implications for further research in an increasingly significant domain.

The paradigm model offered as the principal tool for grounded theory analysis (Strauss and Corbin, 1990, p. 99) requires detailed consideration of the phenomenon in question in relation to causal, contextual and intervening conditions, action/interaction strategies and, finally, consequences. As outlined in Chapter One, this model relates the various components to offer a unified and integrated whole.

**(a) PHENOMENON or CORE CATEGORY**

This chapter first considers the core category, or "phenomenon" for this study: mathematical software use. As a result of the analysis of data from the various respondents, it is now possible to offer a detailed description of the nature of such use, in which various contributing factors define its frequency and form. This form is recognised as composed of quite distinct dimensions, ranging from *non-use* and *passive use* at one extreme, to *strategic use* at the other. As previously noted, this phenomenon of *strategic software use* occupies a position of central concern in the present study, representing as it does a powerful and desirable condition for learning.

**(b) CAUSAL CONDITIONS :**

Having defined the central phenomenon in terms of its specific dimensions, it may now be situated in relation to the key *causal condition* which defines its occurrence, a cyclic framework which relates the mathematical situation, its interpretation and subsequent action on the part of the learner, and the subsequent evaluation of the result of this action. This places the tool use in relation to both user and learning environment.

**(c) CONTEXT**

The nature of the tools themselves and of the learning environment provide the *contextual conditions* under which the phenomenon takes its specific form. It is possible now to identify those features associated with "good" mathematical software in the context of the algebra learning experiences encountered in this study. It is possible, too, to identify desirable features of the learning environment, under which conditions, *strategic software use* is considered most likely to occur.

**(d) INTERVENING CONDITIONS**

Aspects of mathematical and pedagogical thinking served as the key *intervening conditions* identified in this study. Of the former, preferred imagery and the extent to which various algebraic forms signalled action strategies on the parts of the users were most significant; beliefs concerning the nature of mathematics and algebra, and the ways in which these are best learned were also identified as critical in determining the extent and form of mathematical software use.

**(e) ACTION/INTERACTION STRATEGIES**

The ways in which the various individuals and groups actually *used* the available software tools was considered in detail in Chapter Eight, and these are here identified as *action/interaction* *strategies*. It is difficult to extrapolate beyond the confines of the present sample, but quite distinct usage patterns were identified, and these may certainly inform the actions and planning of other practitioners, and perhaps form a basis for subsequent research. It *is* possible to identify two ways in which the available tools (especially computer algebra tools) were found to be most effective in this study: as support for extended mathematical manipulative processes (such as equation solving and completing the square), and as support for investigation and exploration of problems and mathematical concepts, freeing the user of manipulative constraints.

**(f) CONSEQUENCES**

Specific positive and negative consequences of the use of the tools in the current context may be clearly identified. Positive results included increased confidence and improved representational repertoires on the part of all participants. At the same time, some evidence was found of misunderstandings and over-dependence on the tools by some of the participants. Consequences must be viewed within the framework of the various contextual conditions already identified.

The network of relations thus created ensures that the subsequent theory is dense in both descriptive and explanatory power, raising the level of abstraction from initial grounding in the data to a well-developed substantive theoretical position. The resultant theory is then considered in the light of related research, and implications for practice and further enquiry.

## The Phenomenon of Mathematical Software Use

Within a given algebra learning context, software use is most likely to take the form dictated by a particular tool type. In the present study, these were primarily:

- Algebra tools (principally for representation and manipulation)
- Graph tools (for representation)
- Number tools (for representation)
- Utility functions (particularly facilities for substituting, solving and calculus available within the *MathPalette* and versatile tools such as *xFunctions*.)

Within these various tool forms, a range of properties has been discerned as defining the nature of the tool use. These were found to include:

- **purpose** (whether for verification of results, for representation, manipulative support, exploration or simply for convenience);
- **goal-directedness** (the extent to which goals were well-defined and achievable, and the persistence shown in working towards these);
- **versatility** (particularly with regard to the use of a range of tools and access to several appropriate representations);
- **confidence** (in both use of the software tools and in the mathematical results obtained);
- **motivation** (both *intrinsic*, resulting from interest and curiosity, or *extrinsic*, resulting from the demands of teacher or assessment).

The specific dimensions, or “levels”, of mathematical software use have already been described in the context of their occurrences within the data. It is now possible to define these dimensions in terms of the properties of tool use given above.

**Strategic** software use may serve a variety of *purposes*, involving at different times all of the categories mentioned above. While open-ended exploration is most readily associated with this level of software use, it also frequently involves verification of results, which is active and often versatile (as the user deliberately and thoughtfully uses available tools as means to validate findings and to support conjecture). Strategic tool use also involves both representational and manipulative actions as mathematical responses.

Strategic use is most clearly defined by its highly goal-directed nature. The selection and use of available tools is deliberate and thoughtful, with clear **intention** to achieve a particular desired end. It is frequently **versatile** in the use of both varied representations and a range of appropriate mathematical and computer-based strategies. Verification of results is commonly achieved through multiple sources. Confidence associated with strategic use is high, both with regard to the mathematical strategies deliberately chosen and with regard to the results achieved, and motivation may be expected to be dominated by intrinsic factors, especially interest and curiosity. While such use may have been initiated from external sources (such as the prompting of a teacher or tutor, or the requirements of an assessment task), without this critical feature of intrinsic motivation, the tool use appears unlikely to exhibit the important element of **persistence**, which appeared as a significant factor within this study.

**Reflexive** tool use appears more limited than strategic use in most regards. In terms of purpose, reflexive use is most commonly associated with verification of results and representation, and least commonly with exploration and manipulation. While goal-directedness may be high in some instances, reflexive use commonly features a lack of persistence on the part of the user, and a limited representational repertoire. In fact, such use was observed most commonly associated with a single representational category - the graph plotter. Confidence varies with such use, from very high to very low, and motivation for such use may be expected to be external, with less personal commitment on the part of the user than was observed for strategic use.

**Random** use of mathematical software may be considered a sub-category of **reflexive** use. It was found only among the preservice teachers who, especially in their early encounters with the software tools, explored the limits of the “zone of free movement” offered them within the confines of the computer modules, and used the tools freely without regard for curricular context, or even any observable goal. Such use, while occasionally versatile, was observed to be low in goal-directedness and persistence.

**Passive** use was most clearly defined by being externally motivated. The extent to which other factors were demonstrated was dependent upon the intervention of the teacher/tutor, rather than the individual user. While confidence may have increased as a result of such use, it was also observed to result on occasion in decreased confidence and lack of understanding, particularly when the tool use extended beyond the zone of proximal development of the student. Such use by the tutor also served to discourage independence and initiative on the part of the students , and so led to limited personal commitment.

**Non-use** of software tools was difficult to examine directly in the context of this study since its occurrence can only be inferred rather than observed. Nonetheless, it is a most significant aspect in terms of understanding the phenomenon of mathematical software use, and must be considered at this point. The most specific instances of non-use of available tools were observed in association with the many review exercises undertaken by the students. Since they had been encouraged both to ensure to the best of their abilities that their responses were correct and to use available tools to assist in this regard, the frequent failure of individuals to do so when answering incorrectly may reasonably be considered as examples of this type of tool use.

Stephen, for example, encountered fifty review questions on topics ranging from beginning algebra and equations to general algebra reviews and the “stress test”. Of these, he answered fourteen incorrectly, but used available tools (computer algebra and table of values) only four times, and these when prompted specifically. Of Ben’s thirty-five review questions, seven were answered incorrectly and levels of confidence frequently dropped to 60 or 70% prior to selecting an answer, and yet these factors were not sufficient motivation for software tools to be used.

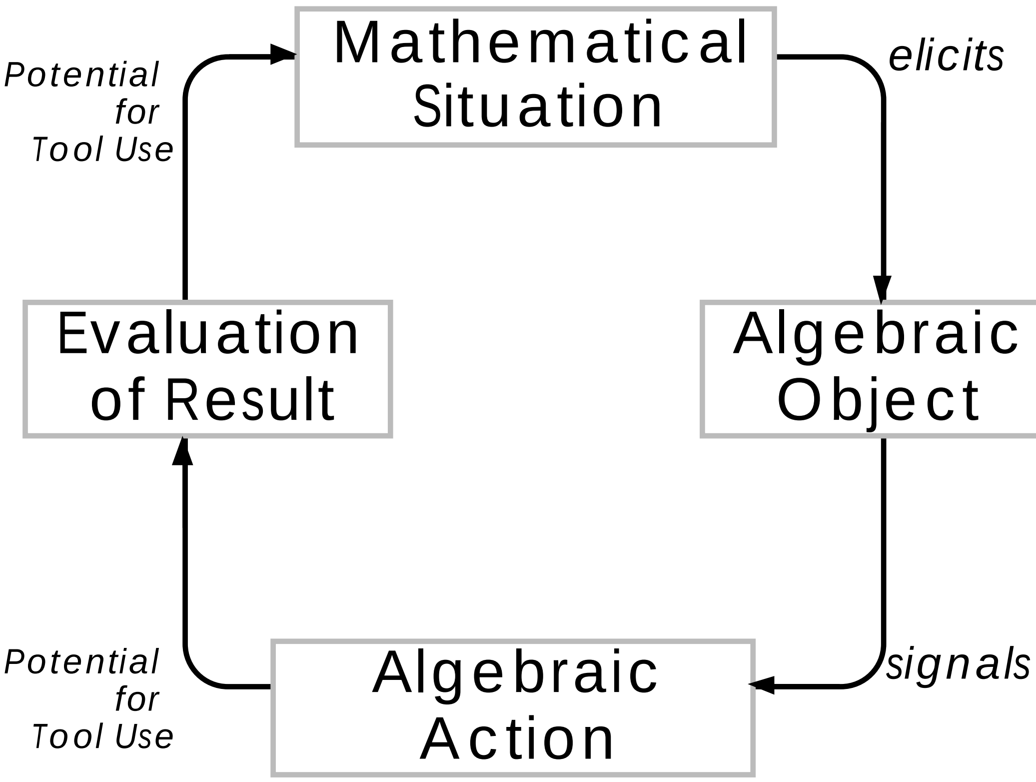
Andrea demonstrated that use of the computer tools did not guarantee a correct result every time - she showed no reluctance to use the available tools when she was uncertain of her result. Of her fifty review questions, she made only five errors since she regularly and on her own initiative verified her results using computer algebra, graph plotter or table of values, whichever she considered appropriate.

It is a primary concern of the theory of mathematical software use which follows to offer some insight into the conditions under which the various dimensions of tool use occurred, and to seek to explain those factors which may have served to both encourage and impede such use. Now that the nature and dimensions of the phenomenon have been detailed, it is appropriate to consider those conditions under which it may be observed.

**Causal Conditions**

Figure 9.1 describes a cyclic framework within which the potential for mathematical software use may be usefully situated. This framework is made up of four components: an appropriate **mathematical situation** in this context is considered to be one which elicits recognition of an **algebraic object** (most commonly an equation, expression, function, graph or table of values). Such an object may be explicit or implied. The former is commonly associated with an algebra learning environment dominated by an instructional perspective, composed of carefully sequenced and deliberate learning activities. Such an environment corresponds closely to the first two **stages of learning** as proposed by van Hiele - the stages of *information* and *guided orientation*.

Figure 9.1: A causal framework for mathematical software use




An implied algebraic object demands both recognition and interpretation on the part of the learner. While such high level cognitive activities may be found at the lower levels of van Hiele's stages, they are more likely to occur within contexts of *free orientation* and *integration*. Failure on the part of the learner to recognise an algebraic object within a particular mathematical situation may not mean that no further mathematical actions can be effected. It does, however, negate the possibility of software tool use within that context, since such use requires an object upon which to act.

Consider, for example, the "unemployment" problem from the module *Something to Think About* which was attempted by both Stephen and Ben. In this problem, information is presented regarding changes to unemployment rates in a hypothetical country over a period of weeks following the election of a new government. While the information is not amenable to use with available computer tools, careful interpretation leads to the recognition of features more commonly associated with curve sketching and calculus, but within an unfamiliar context. Once recognised, the activity invariably resulted in insights regarding, not only the applications of calculus to curve sketching, but also as to the nature and purposes of the important concept of the derivative as a rate of change. This was a rich mathematical exploration which did not require software use, but certainly involved mathematical actions and thinking.

Recognition of an algebraic object may be considered a condition which is necessary but not sufficient for the occurrence of mathematical software use. The object itself, then, must signal a **mathematical action** from the repertoire available to the individual learner and *the nature of the object as perceived by the user will influence the way in which it functions as a signal to act mathematically*. Such a repertoire will contain traditional algebraic actions (simplify, expand, factor, solve, substitute, sketch, differentiate or integrate). Within the technology-rich learning environment created for this study, however, all of these actions were also available using software tools, in addition to representational actions enhanced (and made possible) by the computer, especially graphing, tabulating, and even animating. The extent to which the individual learner has **integrated** both traditional and computer-based mathematical actions must be considered a critical feature in the use of software tools. The potential for tool use at this point is largely dependent upon the extent to which such integration has occurred. Of the students, only Andrea appeared to display such integration, choosing freely from both traditional and computer-based approaches to given mathematical situations. While the level of integration must clearly be influenced by the algebraic thinking of the individual, the results of this study suggest that factors associated with pedagogical thinking (attitudes and beliefs concerning algebra and algebra learning) were far more influential as determinants in the use of available tools.

Having recognised an algebraic object within a given mathematical situation, the learner then chooses from a range of available actions (which may or may not involve the use of software tools). Such action produces a result which must be **evaluated**, usually in terms of an expected outcome. *It was common at this point for students to use available software tools for purposes of verification of results which had been obtained by traditional means.* Potential for tool use at this point was high, as the use of the computer for purposes of verification of results appeared to be perceived generally as both helpful and legitimate, in contrast to the use of the software to support and replace traditional approaches.

Traditional mathematical actions as observed in this study tended to move relatively quickly to a point of closure. Algebra was commonly associated with obtaining an “answer”, usually through the application of a well-defined sequence of steps. Such a perception is seen as largely incompatible with a focus within the learning environment upon open-ended exploration. In fact, the readiness with which even high ability students such as Stephen would conclude their computations, even while expressing less than full confidence in their results, was noted as a source of some concern. When limited to traditional methods, then, the stage of **evaluation** appears likely to conclude the mathematical process with brief verification using whatever means are available.

Use of computer tools, however, served to encourage a cyclic aspect in this process. Even when used only for verification of results through use of an alternative representation (such as viewing a graph to check the solution of an equation), the user is presented with what is effectively a new mathematical object or situation, requiring further interpretation and the possibility of subsequent action. If the user exhibits those characteristics associated above with strategic software use (goal-directedness, versatility, perseverance and, most importantly, curiosity) then the stage of evaluation may be expected to lead to a new sequence of mathematical action, interpretation and reflection, and such was observed frequently within the data.

Figure 9.2: Mathematical software use situated within a causal framework

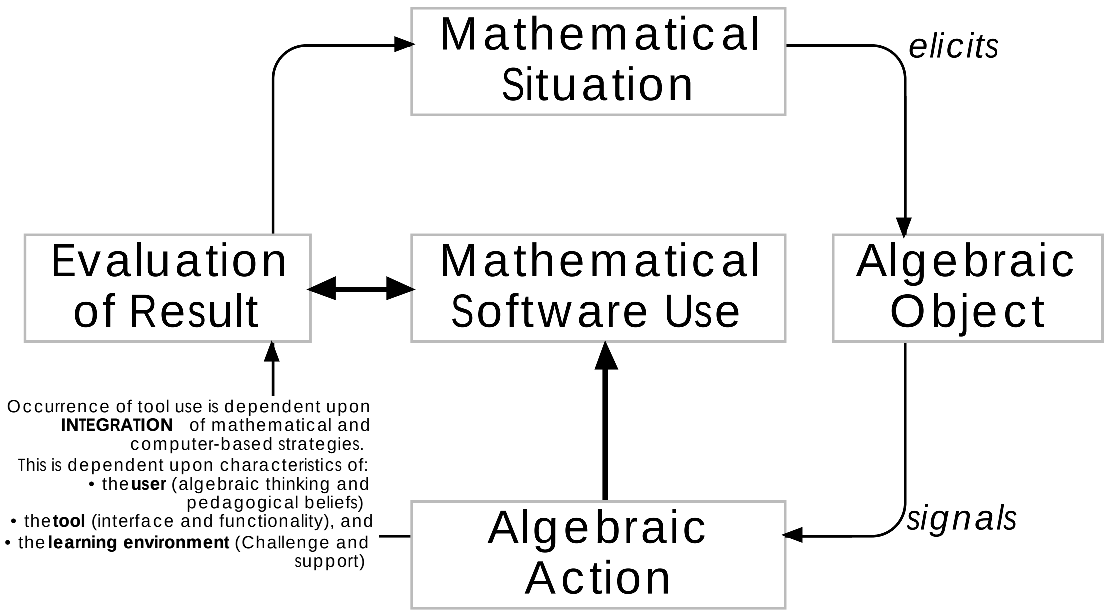


Figure 9.2 situates mathematical software use within the causal framework described. Both the stages of *action* and *evaluation* are likely to give rise to tool use, the latter in a potentially cyclic way. The potential for tool use, however, is limited by the degree of *integration* present on the part of the user, dependent upon characteristics of user (algebraic and pedagogical thinking), tool (interface and functionality) and learning environment (the balance between challenge and support).

No single factor, then, can guarantee that mathematical software use will occur. Rather, several key aspects of individual thinking play important roles in deciding this phenomenon, and these will be detailed as intervening conditions in this theory. Prior to this, however, it is necessary to examine the role of specific **contextual conditions** within this process.

**Contextual Conditions**

The context within which mathematical software use occurs is considered in this study to be dominated by those factors associated with both the mathematical software tools themselves, and the algebra learning environment within which they are available. Both factors served to define and direct the interactions of individuals with mathematical tools.

With regard to the software tools themselves, the twin features of **interface** and **functionality** appeared to figure strongly in determining their use by both students and preservice teachers. Simplified entry of algebra forms appeared to encourage such use - even the more experienced users commonly made errors of entry, typing 4x instead of

4*x when required. It was clear that the algebraic form required by the software should mimic as closely as possible the usual written form. Thus, 4 - 3x is preferable to 4 - 3*x, and $x^2$ - 4x + 4 is to be preferred to x^2 - 4*x + 4. All participants adopted the use of the option key for the placing of exponents quickly and easily, providing simple access to the two-dimensional formatting which is the norm for algebraic forms.

The creation of the "palette" as a means of simplified entry of algebraic forms was an important aspect of the development of the algebraic learning environment for this study. This method is used by such quality *Macintosh* applications as *Theorist* and *ANUGraph*, and appeared to offer an ideal means by which quite difficult algebraic forms could be entered without recourse to specialised code or instructions. The overall response to the palette, however, was disappointing. Although participants used it when prompted, it was generally found to be both cumbersome and slow, and keyboard entry (in simplified form) was invariably the preferred option. Particular problems were found in using the palette to create complicated expressions, especially those involving fractions and exponents. Although improvements were made in response to observed difficulties (such as automating the closing of parentheses), the palette as a form of algebraic entry would need to be far more intuitive than appears to be currently the case for it to be preferred over simple keyboard entry procedures.

The functionality of the various software tools is, in some respects, a complementary issue with interface. A significant factor in the strong preference shown by all participants for the graph plotter as a mathematical tool must be the fact that it does a single job well. Tools which offer a wide range of mathematical choices are likely to act to increase uncertainty in student users. As Goldenberg (1988b) observes regarding multiple representational software, “(w)hile potentially reducing ambiguity, multiple representation also presents a student with more places to look and is potentially complicating and distracting” (p. 136). The same may be applied to much available computer algebra software, frequently offering hundreds of potential choices for action. At the same time, a broad range of functionality is a useful feature, and so the critical factor appears to be the **accessibility** of the various features. The range of available functions should be clearly visible to the user and simple to access. Thus, programs such as *CoCoA* which require specific command-line instructions and offer the user a blank page and no useful menus from which to access commands must be considered a poor choice. Even the *Theorist* interface fails to support access to the full range of available functions in a way which is intuitive to students.

As noted previously, the interface of the program *Calculus T/L II* appears to satisfy this demand most effectively, making available those functions appropriate to the current algebraic object, and so actively reducing levels of complexity and uncertainty. It is unfortunate that this program requires entry of algebraic forms in unsimplified format.

The development of the *ToolKit* menu across all instructional modules was a deliberate attempt to reduce uncertainty and provide access to available mathematical functions. In this way, algebraic forms could be entered in simplified form (or using the palette if desired), and then pasted into other software tools which are linked through the *HyperCard* interface. While *xFunctions*, *Theorist*, *MathMaster* and other commonly used software tools may each require entry in a different form, students using *Exploring Algebra* were provided with a simple and consistent format for entry of algebraic forms which could then be pasted into any of the available tools.

The evidence of this study suggests certain features which must be considered desirable in algebraic software. The **interface** must be clear and intuitive, with available functions clearly visible and easily accessible. This is most readily achieved through the use of pull-down menus which list available mathematical actions (and ideally such actions may be accessed through visible on-screen buttons as well).

Entry of algebraic forms must be simple and closely approximate written forms. The addition of some intuitive method for entry of exponents (such as the option key or the "up arrow" key) is preferable to the use of computer characters, such as "^". Display of the algebraic form must utilise full two-dimensional formatting, allowing users to verify that they have entered the desired expression correctly.

Both graphical and tabular representations must be open to manipulation, permitting adjustment of all parameters in addition to quick and easy facilities for "zooming in" and "zooming out". Axes must be clearly labelled for graphs, and options should be available both for grid lines and labelling in multiples of $\pi$. Flexible entry of algebraic forms appears to be desirable as a means of encouraging versatile thinking regarding algebraic objects. Thus, while an equation such as "$y = 2x - 1$" may be the preferred form for both graphing and tabulating, it should be possible to enter alternative forms such as "$2x - y - 1 = 0$" and even expressions such as "$2x - 1$".

Manipulation of algebraic forms should be under the control of the user, while supported by the software. If an equation is being acted upon, for example, the program should automatically act upon both sides, reinforcing and supporting traditional methods. A record should be visible of each step of the interaction, allowing students to follow the process by which their result was achieved. This computer-based support and display of each step of an algebraic process was considered by the students in this study to be the most useful feature of the computer algebra software which they used. At the same time, there were frequent occasions within the tutorial situations in which a quick result was desired in order to verify a computation. At such times, a computer-generated result encouraged both verification and further exploration. For this reason, *ToolKit* facilities were added for a range of common mathematical processes which involved quite complex computations, including equation solving, derivatives, areas under curves, and even loan repayments. Having ready access to such features permitted strategic use of the software as a convenient means of checking both results and conjectures.

It appears that “good” algebra software should support both the development of algebraic processes and convenient access to a variety of algebraic results. It should offer at least symbolic, graphical and tabular representations, and facilitate movement and transfer of information between these. Access to the various mathematical functions of the software must be clear and intuitive in order to minimise uncertainty and to encourage integration of computer-based mathematical actions with more traditional methods. Above all, the user must feel “in control” of the software, not controlled by it. The ease with which, for example, the user may return to a previous line and edit the contents rather than retyping demonstrates such a level of control.

The nature of the algebraic learning environment must also bear strongly upon the use of available mathematical tools. The results of this study suggest that, in order to encourage strategic tool use, the environment should be challenging and open-ended. Highly sequenced and predetermined instructional programmes are analogous to teacher-dominated classrooms - they tend to stifle initiative and curiosity, and reward task completion at the expense of enquiry and exploration.

An important result arising from this study concerns the availability of software tools: it appears that such tools can be *too* available under certain circumstances. High incidence of reflexive tool use appeared clearly linked to the *hypertext* design feature in which it was possible to access the graph of any algebraic form encountered in the instructional modules simply by clicking on it. This feature appeared to encourage a superficial viewing of the representation, and, frequently among both students and preservice teachers, an automatic response to moving through the program. In order to encourage more active participation, users should actually enter each algebraic object themselves, and then act upon it in whatever way they choose. As mentioned previously, this **reconstructive** act is likely to force a more analytical viewing of the algebraic object under consideration, and to actively discourage the superficial and passive approach observed commonly in relation to reflexive tool use.

There were other respects, too, in which the design and nature of the computer tools themselves may have contributed in a negative way to student learning and understanding of key mathematical concepts, particularly those associated with the domain and range of functions. The graphing utility developed for the project (based upon a simpler tool created by Dr Khoon Yoong Wong of Murdoch University) was a powerful and versatile package, but was unable to correctly plot discontinuous functions. Thus, single point discontinuities (such as that across the origin in the hyperbola xy = 1) were joined by a line from the bottom of the screen to the top. Graphs of such important functions as **y = log(x)** and **y = $\sqrt{x}$**, which are undefined for negative values of x, actually plot the value x = 0 along the x-axis in this undefined region. This aspect of the technology was not noticed by the respondents, but may have contributed to subtle misunderstandings. The same misunderstandings may have resulted from the activity in the *Beginning Algebra* module in which the relationships between family members were discussed as an illustration of the function concept. While the relationship "is the wife of" was identified as a function in the mathematical sense, the program failed to draw attention to the important role of domain and range in this context. In particular, it was overlooked that, for the domain in this case (the members of the family), the relationship is undefined for all but one member. Although such an approach to this important mathematical concept is appealing, it is now recognised that it is fraught with dangers and likely to cause confusion and subtle misunderstandings. The tools and the nature of the learning environment itself must be mathematically correct in all respects if they are to be effective in building firm foundations for further study.

Mathematical situations within a technology-rich learning environment should serve to stimulate enquiry and exploration, in addition to discussion and cooperative strategies within social learning contexts.

Van Hiele's third stage of learning, *explicitation*, specifically demands verbalisation as a means towards achieving cognitive progression, in the same way that Vygotsky's theories place social interaction at the heart of effective learning. In this regard, the computer plays a particularly significant role for algebra learning, since it makes explicit both the objects of attention and the processes by which these are acted upon. By making public algebraic thinking and action, mathematical software tools uniquely encourage shared meaning among co-learners, and support insightful evaluation of student thinking and understanding by their teachers.

Finally, a technology-rich algebra learning environment should be characterised by versatile thinking about algebraic ideas using multiple representations, with explicit attention directed towards developing active and meaningful links between these. Such thinking may be encouraged through the thoughtful use of open-ended tasks which appear accessible to the students and yet offer challenges which suggest the use of appropriate tools. For teachers, "learning to ask new questions" remains a critical aspect of "learning to use new tools". The role of the teacher within such an environment must be a flexible one - encouraging individuals to go beyond their present capabilities, and yet not allowing them to become too dependent upon their scaffolding tools. The zone of proximal development remains a concept central to an understanding of such a learning environment, one in which "the only 'good learning' is that which is in advance of development" (Vygotsky, 1978, p. 89) and where it is accepted that "what a child can do with assistance today she will be able to do by herself tomorrow" (Vygotsky, 1978, p. 87). Such a view appears unusual within the context of algebra learning, where traditionally the greatest effort has been placed upon the acquisition of manipulative skills by individuals working alone and unaided. The technology-rich learning environment defined within this study is characterised by two essential features: **challenge** and **support**. The tension between such a view of learning and traditional approaches has led to the identification of a mathematics learning culture which, with aspects of algebraic thinking, may be considered as *intervening* conditions within a theory of mathematical software use.

### Intervening Conditions

Both mathematical and pedagogical thinking function as intervening conditions with regard to mathematical software use within an algebra learning situation, potentially impeding or encouraging such use for different individuals.

This study clearly demonstrates that a given algebraic object may be perceived in a variety of ways and associated with a range of action strategies. While simple linear and quadratic equations and graphs signalled predictable responses, simple expressions and tables of values proved more difficult to interpret. Both students and preservice teachers responded to what may be seen as a powerful drive towards closure related to their algebraic actions, a drive frustrated by such a simple expression as **4 - 3x**. When Sfard and Linchevski (1994) asked what individuals see in an expression such as **3(x+5) + 1** (p. 191), they examined the results in terms of Sfard's theory of reification, involving a conceptual move from viewing mathematical concepts as procedures to viewing them as objects, capable of being acted upon in their own right. This study examined simpler objects and found similar perceptions, but focused more closely upon the signal character of such objects and the subsequent repertoire of mathematical actions which they elicited.

Familiarity with graph plotter and table of values served to increase this available repertoire for all participants, offering at least two new strategies for use with what was found to be an impoverished algebraic form. Computer algebra software, however, offered no such addition, merely an alternative approach using traditional methods supported within a computer-based context. The ability of students to integrate the two approaches may be recognised as a determining factor in the use of algebraic manipulation software. This study suggests that such integration may begin early in the formal study of algebra, framed within meaningful context and following upon quite extensive use of the tabular representation. The manipulations of algebra must be grounded in numerical understanding.

The selection and use of available mathematical software tools, then, will be influenced by the algebraic thinking of the user. Algebraic forms which traditionally have signalled a graphical representation (such as the form **y = 2x - 1**) were found in this study to trigger the use of graphing software frequently and spontaneously by all participants. Although the table of values was frequently described as very helpful, it remained a subservient representation to the graph, probably because of difficulties encountered in interpreting tabular information which have been reported elsewhere. Ryan (1993, p. 369-370), citing work by Herscovics (1989) on cognitive obstacles and Yerushalmy (1991) on multiple representational computer software, observes that particular problems arise from the use of multiple representations and the table of values in particular. These include over-reliance upon a single representational form and the assumption that students will naturally and spontaneously “make connections” between different representations. An extensive study by MacGregor and Stacey (1995) involving over 1200 students in two Australian states further supported findings which indicate that tables of values present quite substantial problems of interpretation and analysis. Particular steps need to be taken in order to build effective skills of intrepretation for this representational form (MacGregor and Stacey, 1995, p. 83). At the same time, the table of values was generally considered a valuable aid to understanding within the current study, and perceived by some participants (particularly Stephen and Tony) as a more flexible tool than the graph plotter, capable of acting upon expressions such as **2x - 1** in addition to the more usual form given above.

Stephen's perception of algebraic forms generally appeared to be dominated by an “input/output” or “function machine” metaphor, an active view involving numerical values being changed according to the functional rule. Such an image influenced, not only his interpretation of tables of values (with which this image identifies most readily), but also his thinking about graphs and symbolic forms. The robustness of this active conception may help to explain Stephen’s cross-representational facility and his general success in algebra in comparison with his peer, Ben, who was strongly influenced by a visual graphical metaphor, which did not transfer easily across representations in the same way as did the function machine image. Dependence upon the graphical form alone appeared to disadvantage Ben in his approach to algebraic problems. Although Andrea displayed evidence of both graphical and input/output imagery. her thinking appeared dominated by the symbolic form, especially the equation. It seems likely that her frequent use of the full range of available software tools contributed towards the cross-representational facility she displayed, but her preference for the symbolic form puts her frequent use of computer algebra tools into perspective, in the same way that Ben's strong tendency to visualise helps to explain his use of the graph plotter. Such interaction between thinking and tool use supports the hypothesis of a recursive relationship between the two offered early in this study.

The use of available tools, then, is influenced by the algebraic thinking of the user, and such thinking may come to take the form associated with preferred software tools. Individual perceptions of a given algebraic object and the repertoire of available mathematical actions which it signals may vary greatly, and so determine the nature and direction of tool use. Critical factors appear to be preferred images of algebra, familiarity with the software tools and the degree of integration of traditional algebraic actions with computer-based strategies. Algebraic thinking, however, must be considered in conjunction with pedagogical thinking regarding algebra learning if variations in software use are to be better understood.

In considering the beliefs and perceptions of the participants in this study regarding the nature of algebra, the ways in which it may best be learned and the role of computers in this process, consistent evidence was found to support the notion of a culture of mathematics learning: a shared set of beliefs and experiences which extended across all groups of participants. This culture served largely as an impeding factor for the use of algebra software, characterised as it was by such features as:

- a view of mathematics as "answer-based", devaluing exploration and open-ended problem solving (those areas in which the software appears most effective);
- a view of algebra as primarily serving a symbolic representation purpose, with little usefulness beyond this role;
- an emphasis upon individual efforts, devaluing both group approaches and the use of external aids (such as computer tools);
- a strong reliance upon individual as opposed to group aids - especially textbooks and hand calculators;
- a dependence upon the teacher as source of knowledge and direction in mathematics learning;
- a limited representational repertoire, dominated by symbolic and graphical forms;
- a lack of reliance upon individual judgement and confidence with regard to their mathematical processes - students appeared quite happy to conclude an answer while expressing little confidence in their result.

Clearly, factors such as these militate against both the *use* and *the perceived need* for open-ended software tools which support and extend mathematical learning. These findings are consistent with the results of other research conducted both in Australia and overseas. Wood and Smith (1993) used a questionnaire adapted from Schoenfeld (1989) to elicit attitudes and beliefs about mathematics from students beginning mathematics and engineering degrees at a New South Wales university. Although such relatively high ability students exhibited positive attitudes and intrinsic motivation with regard to mathematics, threequarters of the sample of seventy four students felt that school mathematics was “mostly facts and procedures that have to be memorised” (p. 594). Alternative approaches to solutions were considered highly desirable (96%) but almost half (44%) felt it was important that mathematics teachers show students “the exact way to answer test questions” (p. 595). Most students felt that a “typical homework problem” should take less than ten minutes (p. 596).

School mathematics in New South Wales is heavily influenced by external examination, and this must exert a powerful effect upon student perceptions and beliefs. A study in progress by Barnes, Clark and Stephens (1995) which compares links between assessment and teaching practices in New South Wales and Victoria has found that teachers in both states value most highly those things associated with high stake assessment. New South Wales teachers (56 teachers in 11 schools) were found to value most strongly “the application of mathematics to real world contexts” and “the use of different mathematical skills in combination” (associated with both School Certificate and Higher School Certificate examinations) and gave low value to “extended and open-ended activities, development of report writing skills, mathematical journals and the provision of substantial written comment on problem solving attempts”. The development by students of investigative skills was rated most highly by Victorian teachers and least highly by those in New South Wales.

The powerful influence of mandated assessment upon teaching practice suggested by this study and the previous “ripple effect” study by Clark, Stephens and Wallbridge (1993) suggests a potential role for assessment in encouraging the use of technology in schools. In the short term at least, the use of mathematical software tools with open-ended assessment tasks appears a potentially useful way in which to introduce the use of technology into classroom practice.

At the same time, the mathematics learning culture observed within the current context is clearly not restricted to New South Wales schools. An extensive study by Garet and Mills (1995) involving almost 400 head teachers of mathematics in the United States examined the influence of the National Council of Teachers of Mathematics *Curriculum and Evaluation Standards* upon curriculum content, teaching practices, use of technology and assessment procedures in first-year algebra courses (equivalent to Year 9 in Australia).

> The data indicate that lecture-discussion and in-class problem sets remain the dominant mode of instruction in first-year algebra... the use of calculators has grown dramatically since 1986... The use of computers, although not as extensive as the use of calculators in 1986 and 1991, is expected to increase substantially by 1996... the use of short-answer tests is not declining and remains the dominant form of assessment (p. 382).

Software use was dominated by graph plotters (54% of departments reported using these), drill-and-practice packages (49%) and exploratory packages for algebra and geometry (45%). Only 29% reported use of spreadsheets, and 21% used symbolic manipulation packages. Although cost and hardware factors were considered significant, the strong preference for graph plotting tools and the relatively minor use of computer algebra is supportive of the findings of the current study. Textbooks remain a major influence upon teaching practice.

When considered within the context of earlier studies of exemplary practice in Western Australia and the United States by Tobin and Fraser (1988), the existence of a prevailing culture of mathematics teaching and learning which was so evident within the present study appears irrefutable. The dominant influence of such a culture upon the selection and use of available software tools, then, occupies a central position within the grounded theory of mathematical software use proposed.

**Action/Interaction Strategies**

The active selection and use of mathematical software tools occupied a central point of focus within this study. Tool-based actions arose in response to the contextual and intervening conditions already considered - the nature and knowledge of available tools, the curricular context and perceptions of the algebra learning environment, and the mathematical thinking elicited by the given situation.

The most frequent mathematical actions for which software tools were used were those associated with graphing (representing), substituting and solving, corresponding to the three main perceptions of the purposes of algebra, as defined by the participants. At a higher level of abstraction, the computer was used most frequently to represent, to verify and, on the part of the tutor, to demonstrate.

Representational actions dominated the computer-based interactions observed within this study. As has already been noted, the graphical form was preferred in most situations, and the students quickly became proficient in the use and interpretation of this software tool. Most effective for this purpose was a “Guess My Rule” game within the IBM-based program *A Graphic Approach to the Calculus*, by David Tall. The computer generated the graph of a selected function type, and students would attempt to identify the particular function. Their guess was graphed, providing immediate feedback. Reluctant at first to make mistakes, the students at all levels quickly overcame this hesitation and learned to use strategic trial-and-error methods to identify different functions. This technique proved so effective that it was incorporated into the *MathPalette* using both graphical and tabular representations. Once again, it encouraged in students familiarity with both representations, and assisted Andrea in particular to become comfortable with the table of values.

Manipulative actions centred upon evaluating substitutions and solving equations. As mentioned previously, the *Theorist* interface encouraged and rewarded both these activities, and a preference was shown by all students for the equation-solving method of moving terms across the equals sign, which was the method supported by this package. While such an approach appears appealing as the preferred method for experienced practitioners, it is also common to find it linked to superficial understanding and rote learning of the solution process, A study by Bell, MacGregor and Stacey (1993) of a group of twenty Year Ten students in a Melbourne school found frequent recourse to what were described as “action memories” in solving simple linear equations. Such memories were tacit, and students were unable to justify their approach. They were also frequently associated with incorrect responses, since the method did not allow for variations in the equation form.

Although the student participants in the present study demonstrated competence in equation solving, both with the computer and without, the method of acting upon both sides of an equation in order to produce a solution would seem pedagogically superior, and computer algebra packages which support this approach preferred. (Note that while *Theorist* does support both methods of equation solving, the manipulation method is so much easier to use that it becomes the preferred option by default.)

As an instructional tool, computer algebra software, then, seems most effective within two situations. The first involves the step-by-step support of extended mathematical processes (such as equation solving in the junior school, "completing the square" in the middle school, and "differentiation by first principles" in the senior years). These processes tend to place high manipulative demands upon students, and so are well-suited to treatment and study using computer algebra tools.

Computer algebra software is also likely to be most effectively used within open-ended mathematical explorations, minimising manipulative barriers and supporting processes of enquiry. As noted within the study, strategic software use occurred most frequently within such a context, in which students were challenged and motivated. Such explorations are not a frequent feature of current mathematics learning situations. However, such use may potentially be encouraged within alternative assessment schemes, in which software tools are made available as a strategic option. The problems gathered and developed for this project are appropriate for use as extended individual and group-based assessment tasks, while actively serving to encourage and demonstrate the use of software tools. As noted in relation to the "ripple effect" studies (Clark, Stephens and Wallbridge, 1993, Barnes, Clark and Stephens, 1995), it is likely to be through such assessment schemes that the use of mathematical software will most readily be integrated into the existing curriculum. Computer tools which make explicit the step-by-step mathematical processes leading towards a solution appear most appropriate for use in open-ended assessment tasks. Word processors have encouraged new approaches to creative writing by making possible a cyclic process involving the refinement of several draft versions. Students using appropriate algebra software may also see their final solution as the end result of an interactive process of refinement, supported by teacher, peers and the software itself.

**Consequences**

Not all the outcomes which emerged from the use of mathematical software tools in this study were intended. Stephen, for example, believed it necessary for a function to be expressed using a particular format: **$f$(x) = 2x - 1** is a function for Stephen, while **2x - 1** is not. Further, **2x - 1** cannot be graphed (although it can be represented in tabular form), while **y = 2x - 1** can. Such misconceptions, while perhaps not serious, arose as a direct consequence of particular features of the software packages which this student had experienced. The use of computer tools with students at all levels, then, must take into account effects such as these. Similarly, Ben's dependence upon the graphical representation must be directly attributed to his encounters with the technology. After becoming comfortable and confident in the use of the graphing tool, Ben became over-dependent upon it.

This project was not intended or designed to establish causality between the use of the software tools and aspects of cognitive functioning by the participants. The experiences of the technology-rich algebra learning environment by both students and preservice teachers were far too limited in relation to traditional mathematical learning contexts to expect clear and attributable changes in skills or knowledge. Even those students who engaged in the research programme for an average of one hour per week over up to two school years spent between four and five times that amount in their usual mathematics classes. It is hardly surprising that the influence of the current mathematics learning culture is so pervasive, and the effects of exposure to an alternative learning environment so few.

Nonetheless, certain consequences of the use of the computer tools could be identified from the data. The evidence of this study indicates strong support for ways in which they may contribute to the learning process, particularly through factors such as:

- increasing confidence in answers (and learning to expect that such higher confidence rates should be the norm rather than the exception);
- increasing the representational repertoire to include tables of values, concrete forms and even animations;
- encouraging exploration and open-ended problem solving by providing tools which facilitate and make possible these approaches.

These three consequences of mathematical software use were widespread among the participants, although varied in degree. Certainly, all student participants indicated improved confidence in their answers as a direct result of the use of computer tools, and all demonstrated some measure of cross-representational facility across symbolic, graphical and tabular forms. All engaged at some stage in strategic use of available tools for the exploration of mathematical ideas. Such use was commonly associated with demonstrations of insight and improved understanding of the mathematical ideas in question.

The responses of all participants reflected positive attitudes towards the use of the computer as an aid to mathematics learning, although most indicated some realistic limitations to their support. Overall, while the graph plotter was enthusiastically accepted as a tool for mathematical learning, the table of values was found to be difficult to interpret at times, and computer algebra software was perceived as being in some ways illegitimate. This is hardly surprising within a culture which convinces students that the “best way to learn algebra” is through repetition, and where the majority of students “prefer to learn the teacher’s method for solving problems”.

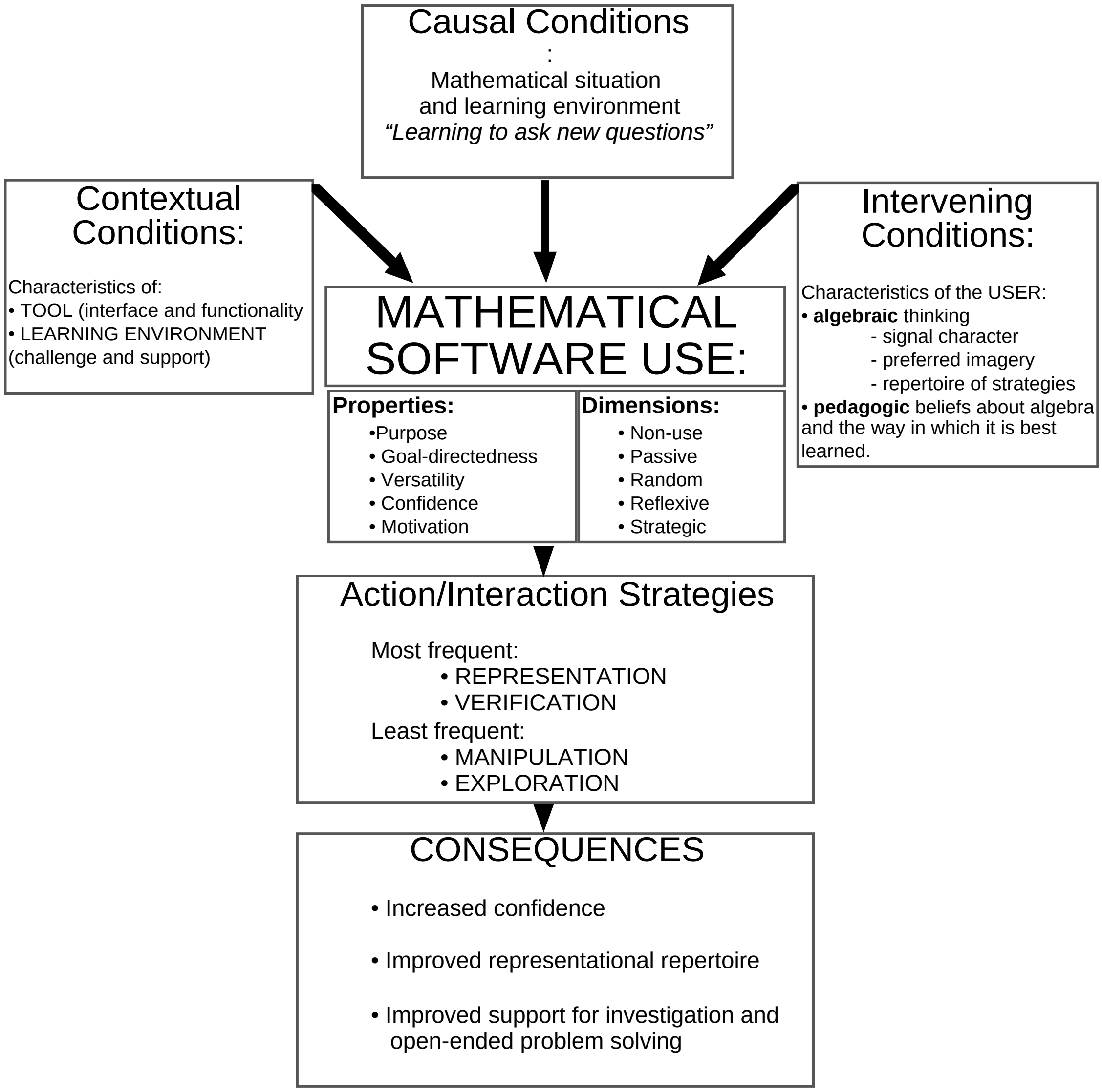
### A Grounded Theory of Mathematical Software Use

The grounded theory of mathematical software use developed through this study situates such tool use within a context of:

(1) Algebraic and pedagogical thinking by the user,
(2) Familiarity with the software and subsequent access to desired functions, and
(3) A learning environment which is both supportive and challenging, encouraging the use of multiple strategies in order to achieve agreed-upon mathematical goals.

Figure 9.3 provides a schematic outline of the relationship between the various component parts of this theory.

Figure 9.3: A grounded theory of mathematical software use: Schematic outline




The learning environment gives rise to a **mathematical situation**, from which the user must elicit a distinct **algebraic object** (most commonly function, expression, equation, graph or table of values). Subsequent use of mathematical software tools is dependent upon recognition by the student of such an object.

The user seeks to act upon the algebraic object in order to move towards a desired state of closure. The **repertoire of available strategies** is dependent upon both the strength of the signal character of the algebraic object, and upon the extent to which computer-based skills have been integrated with more traditional methods.

The action strategy taken at this stage (which may or may not involve computer use) produces a result which must be evaluated in terms of the extent to which it brings the problem situation closer to closure. While the influence of the prevailing culture of mathematics learning is likely to act against the use of software tools to assist manipulative actions, tool use for purposes of verification of results appears not to inspire the same resistance. Representational tool use, too (especially involving the graphical representation), appears to complement existing practice, while manipulative use directly confronts traditional approaches.

Under conditions of strategic software use, the evaluative act is likely to present to the user a new mathematical situation, requiring further interpretation, action and reflection. Strategic use is characterised by persistence, curiosity and the use of multiple strategies for both exploration and evaluation of results. It frequently accesses multiple representations and uses a range of available software tools. Such use is goal-directed, flexible and frequently insightful.

Under conditions of high availability of tools, high demand for task closure and extrinsic motivation, the use of available tools is likely to be **reflexive**, as the user selects quickly and makes superficial use of desired functions. Such use is commonly associated with use of the graph plotter as the preferred tool of choice and an environment which rewards results rather than process. Reflexive tool use may be discouraged by requiring active involvement on the part of the student, particularly in the reconstruction and entry of algebraic forms. Students must be participants, not observers, in the mathematical process.

Under conditions of free orientation, **random** tool use may occur, as the user experiments with available tools. Such use may be versatile, but it is low in goal-directedness. **Passive** tool use occurs under conditions of an imbalance of power between multiple users. While such use commonly involves teacher (or tutor) and students, it may also be observed between peers working together. The user in such a context hands over the responsibility for learning to the active participant, and is likely to gain much less from the experience than would an active participant. This role is associated with the use of the computer as a tool for demonstration (Ganguli, 1992).

Finally, there are conditions under which tool use is appropriate, and yet no such action is taken. It was common for participants to express lack of confidence in their answers, and yet to take no action to validate or disprove their result. Students appeared to feel no personal commitment regarding their involvement in algebra learning: motivation is extrinsic and the demand for closure apparently far exceeds interest or curiosity regarding the mathematical situation.

Within the constraints of the research design, the grounded theory proposed offers the possibility of prediction regarding the likely use of mathematical software tools and the encouragement of strategic use within algebra learning situations.

The principal uses of computer tools for mathematical purposes within this study were found to be for **representation** (using graph plotter and, less often, table of values) and **verification** of results. Although well-suited to support open-ended investigation, such use is likely to remain rare under the influence of a culture of learning which rewards closure and identifies algebra with "finding an answer" using automated and predetermined action sequences. Consequently, computer algebra tools may best be introduced into the current mathematics curriculum in two ways:

- As means of supporting students in the learning of sequential mathematical procedures (such as equation solving in the early years). Computer tools which both support and make explicit the process provide a useful aid in such areas.
- As tools for supporting open-ended assessment tasks, and so encouraging and motivating mathematical enquiry.

The evidence of this study suggests that teachers may encourage strategic software use through the creation of a learning environment within which:

- *students are comfortable with the available software tools*. The interface should support ease of entry of mathematical forms and make the range of mathematical functions clearly available.
- *mathematical tasks lie within the zone of proximal development of the students*. Students must perceive the task as potentially

achievable, although beyond their present capabilities unaided.

- *students must be able to elicit from the task a mathematical object* which is capable of signalling appropriate action strategies involving the integration of mathematical and computer-based actions.
- *open-ended investigation is perceived as a valid means of achieving a solution*, which may be only one of several appropriate responses to the task.
- *The use of multiple strategies for verification must be perceived as a necessary component of mathematical enquiry*.
- *students must be motivated*: persistence and some measure of personal commitment to the solution process must be evident.

The strategic use of mathematical software tools is indicative, not only of a high level of computer-based competence, but of insightful and strongly connected mathematical thinking. Conditions under which such use may be encouraged should be a feature common to all mathematics learning situations.

## Conclusion: Impediments, Imperatives and Implications

The grounded theory proposed appears both dense and integrated. As a teacher learning to use new tools, I feel confident that the initial demands of my action research enquiry have been satisfied. The phenomenon has been examined in great detail, and situated within a broader context: one involving images and definitions of algebra, perceptions and beliefs about learning, recognition of the characteristics of "good" algebra software and some appreciation of what a "technology-rich algebra learning environment" may look like. As the researcher and prime motivator for this study, I feel confident in my new knowledge and skills regarding "teaching with these new tools". Always there is more to know, but I recognise that at least now this teacher knows enough to "get started".

One of the most informative features of the study involved recognising the formidable array of impediments to the use of mathematical software tools for algebra learning. While readily recognising what might be termed "institutional" constraints (particularly lack of access to appropriate hardware and software) this study made it obvious that the real impediments were buried deeper, within the psyche of mathematics teaching as it has been practised in our society for one hundred and fifty years. This impediment will be difficult to overcome, since it arises from perceptions of the very nature of algebra, as it is found in schools.

For every impediment associated with the use of computer technology in schools, there are a growing number of imperatives. From the demands of society for a technologically-literate and mathematically competent work force to the surprising wonders of chaos theory, the symbiotic disciplines of mathematics and computing will continue to cross paths again and again. As computer technology becomes ever more accessible, appropriate and powerful as an environment for learning, the possibilities it offers for improved understanding of mathematical ideas and support for mathematical skill development become impossible to ignore.

This study may be seen to have implications for a variety of audiences with an interest in the role of technology in mathematics learning. For those interested in examining further the issues related to this critical field of enquiry, the theory proposed here suggests many new questions. At an individual level these include the effects of different computer algebra tools upon the development of manipulative skills and mathematical understanding, a more detailed examination of the role of preferred imagery in algebra learning and its relationship with action strategies, the transfer of meaning across representations and the derivation of meaning from new representations made possible by computer technology.

With the increasing power and availability of hand-held computers and graphic calculators, issues of personal access to technology must also be addressed. Students in the current study used their own calculators effectively and often. As Smith (1992) proposed, the distinction between a *Social Constructivism model* of tool use (in which the tool and the user act jointly upon the mathematics "out there") as opposed to an *Individual Constructivism model* (in which both mathematics and tool are "out there") is relevant here. The calculator fitted comfortably within the "personal space" of the user; the computer did not. In this study, it remained "out there" and students failed to achieve the comfortable and spontaneous familiarity with the computer tools which distinguished their calculator use, even after protracted experience. It remains to be seen whether hand-held computer tools capable of supporting algebraic manipulation and multiple representations may be more readily accepted and utilised.

At a classroom level, factors influencing the classroom use of computer tools for mathematics learning must be considered, especially the role of the computer in assisting group and cooperative approaches by making public both algebraic objects and processes. The physical impediments associated with access to the technology for large groups and the changing role of the teacher within a technologically-rich learning environment are increasingly important considerations. Asp, Dowsey and Stacey noted that teachers moved from an instructional to a management role and tended to miss capitalising upon learning opportunities as a result (Asp, Dowsey and Stacey, 1993, p. 53). At both classroom and individual levels, the responses of teachers to technology remain of critical importance. The decision to restrict the present study to those engaged in algebra learning was a deliberate one, allowing a necessary restriction of focus which would not have been possible otherwise. Nonetheless, the next logical step from this study must be an examination of the individual interactions of teachers with the technology, and a close examination of the nature and influence of their algebraic and pedagogical thinking upon tool use. Such a study offers much in deepening and potentially verifying the present grounded theory of mathematical software use.

Broader issues still relate to the influence of technology upon the nature of learning and instruction. Trying to adapt the use of the technology to fit the existing mathematics classroom may well be a retrograde step: as demonstrated forcibly in this study, there are fundamental incompatibilities between the effective use of technological tools and the prevailing culture of mathematics learning and instruction, at least as evident within this sample. If such tools are to be used to their full advantage, then critical beliefs and assumptions about the nature of algebra and the ways in which it is best learned, and even of what constitutes effective learning and successful teaching must be revised and, perhaps, sacrificed.

For mathematical software developers, this study provides detailed information by which software tools for mathematics learning may be evaluated. This study provided an opportunity to gather, use and evaluate a large collection of mathematical software tools. Very few of those currently available satisfy the criteria developed through interaction with the participants. There remain opportunities for the creation of appropriate mathematical tools which support and encourage mathematical enquiry, and potentially offer access to much of mathematics which is interesting, relevant and important, but is currently denied to the majority of students who “do not possess adequate algebra skills”.

For teachers of mathematics, this study provides an opportunity to share in the learning experience of a colleague. Before it is possible teach effectively, it is necessary to have some understanding of the ways in which individuals learn, and this study specifically provides such information with regard, not only to the use of computer software tools, but also in relation to the ways in which individuals perceive and act upon algebraic forms, the influence of preferred algebraic imagery, and beliefs and perceptions regarding learning. Further, the computer-based instructional modules developed, trialed and evaluated within this project provide practical ways in which teachers and students may explore the use of computer tools for the learning of algebra. For teacher educators, too, the modules provide a simulated algebra learning environment in which preservice teachers may examine alternative approaches to algebra teaching and learning made possible through the use of software tools.

This study offers powerful arguments for the use of appropriate computer technology in the creation of an algebra learning environment which emphasises meaningful contexts, the development of versatile thinking about algebra through multiple representations, and a balance between challenge and support. Mathematical software tools offer unique opportunities for the development of algebra skills within a context of improved understanding and active involvement by students in their own learning. By making explicit both algebraic thinking and processes, appropriate software encourages feedback, verbalisation and cooperative approaches on the part of the learners, and supports informed evaluation by the teacher.

Most important, however, is the possibility for exploration of mathematical ideas supported and made possible by such tools. The view of mathematics which follows from such an approach is one which is vibrant and exciting. Not only are learners placed in positions of responsibility and control regarding their own learning, and the likelihood of learning with true understanding substantially heightened, but, for perhaps the first time, teachers and students potentially become co-learners in stretching the boundaries of their discipline.