Two
Review of the Literature

> First we must see how the teaching and learning of traditional topics can be improved with the full use of technology. Can these topics be taught better, faster, and with greater student understanding? When this question has been answered, curricular issues can be addressed.
>
> (Waits and Demana, 1988, p. 332)

Seymour Papert (the creator of *LOGO*) dreamed of the computer as providing the basis for a “mathland”, “which is to mathematics as France is to French, where children would learn to speak mathematics as easily and as successfully as they learn to speak their native dialect” (Papert, 1980, p. 230). Over a decade later, his dream remains far removed from the realities of the vast majority of schools and classrooms. The reasons for this lack of impact are diverse. They include concerns associated with computer technology, such as access, availability and cost, appropriateness and ease of use of both software and hardware. They include also characteristics of the teachers and learners who would use mathematical computing technology - from traditional ways of viewing and doing mathematics to the difficulties of bringing about change in schools and classrooms.

The review of the literature which follows seeks to provide an overview of issues associated with incorporating technology into the related processes of teaching and learning mathematics. In so doing, it establishes a clear direction for the course of the project which follows. The major focus in this study is the use of advanced mathematical software (particularly computer algebra software) by students and student teachers, and the interactive relationship between thinking and action in this context. In order to bring this broad area into a sharper perspective, responses of the participants will be analysed using the SOLO taxonomy (Biggs and Collis, 1991, Collis and Biggs, 1991) and the van Hiele model (1986). These methods are intended to make explicit both modes of representation of algebraic ideas and levels of thinking within an algebra learning context.

Ways in which the technology may contribute to improvements in the development of key concepts such as “algebra”, “equation”, “function” and “variable” will also be considered, particularly in the light of recent work in this area by Sfard (1991, 1992, 1994) and more established work by Bruner (in Bruner and Anglin, 1973). An overarching influence in the analysis of individual interactions with the software tools is that arising from the work of Vygotsky, particularly as it has been applied and extended to the description of collaborative and tutorial learning situations by Bruner (1968) and, more recently, Wood (1980, 1986). Since data collection for this project occurs as a result of individuals interacting with the software in tutorial or collaborative situations, these “scaffolding” studies are most appropriate.

Computer Algebra Software as a Tool for Learning

The potential for computer algebra systems to enhance the teaching and learning of mathematics has tantalised educators now for more than a decade. Some seventy articles and dissertations dealing specifically with computer algebra have been identified in twenty different journals and numerous books; the vast majority of these have been published in the past three years. Of these, however, only nineteen describe the results of research in this area and only six studies involved secondary students (Boers, 1990, Rosenberg, 1990, Sheets, 1993, Wood, 1991, Yerushalmy, 1991a, 1991b) . The increasing volume of publications in this field points to a growing interest and awareness of the possibilities for the use of such powerful tools; the lack of research evidence as to the effects of such use points further to a growing need in this regard.

Computer algebra has been available for personal computers since 1980, in the form of *muMATH*, a powerful symbol manipulator which, through innovative design, was capable of performing exact arithmetic, calculus and much of the symbolic algebra required from school through university - all using only 64 kilobytes of random-access memory! Compared to modern computers which frequently access more than 4, 000 kilobytes, this was, even by present standards, a remarkable achievement. Inexpensive (at $US40), versatile and powerful, *muMATH* stimulated enormous interest among groups of mathematics educators who saw it as potentially capable of influencing mathematics curricula away from the repetitive focus upon manipulative skills which had so characterised instruction in the past

(Fey and Heid, 1984, Fey and Good, 1985, Coxford, 1985, Ralston, 1985). This theme was pervasive through both the National Council of Teachers of Mathematics' Yearbooks in the middle of the decade, *Computers in Mathematics Education* (Hansen and Zweng, 1984) and *The Secondary School Mathematics Curriculum* (Hirsch and Zweng, 1985), and recurred again in the 1988 Yearbook, *The Ideas of Algebra, K-12* (Coxford and Shulte, 1988). *muMATH* provided the focus for numerous publications which described its potential for enhancement of such diverse areas as engineering analysis (Lance, Rand and Moon, 1985), secondary school algebra (Beard, 1989), solving differential equations (Mathews, 1989b), teaching the fundamental theorem of calculus (Mathews, 1989c) and, of course, teaching general calculus (Freese, Lounesta and Stegenga, 1986, Mathews, 1988, 1989a).

For all its power and affordability, *muMATH* has had a negligible effect upon mathematics classrooms in secondary schools. This is partly because it was not easy to use (with an enormous vocabulary of commands which required frequent reference to the manual) and it had no graphics capability; teachers also had no clear guidelines about the use of such tools. The inherent difficulties and essential inappropriateness of the program for use in schools were sufficient reasons for teachers not to persevere with its implementation.

In recent years, *muMATH* has made way for a new generation of open-ended symbolic manipulation software, equipped in most cases with such essential features as graphics capabilities, true mathematical formatting, intuitively simple command structures and extensive mathematical capabilities. Such tools are referred to here as "enhanced computer algebra tools", and include programs such as *Mathematica, Maple V*, *Derive, Theorist, MathCAD, Milo* and, here in Australia, *SymbMath*. These tools vary widely in their costs and capabilities, but all are capable of performing the majority of algebraic manipulations and algorithms required for the secondary school and beyond. Less powerful software, such as *CC - the Calculus Calculator, CoCoA* and *MathMaster* also provide access to many of the same functions as the more powerful programs of this type, but at minimal cost.

In seeking to explore the use of computer algebra software in mathematics teaching and learning, cost is but one of a number of relevant factors. The lack of impact of such tools over the past decade needs to be viewed in terms of both hardware and software constraints, in addition to issues of access and availability of computing technology in schools.

Computer algebra is extremely intensive in its memory demands. The complex algorithms and numerous calculations required for computers to carry out symbolic manipulation have generally required computing power beyond the reach of most schools. The rapid increases in both available memory and raw computing power of the past few years, however, now see affordable computers capable of operating such software. The release of low-cost computers for both Macintosh and MS-DOS operating platforms means that the hardware required for most of the existing computer algebra tools is now reaching schools and classrooms.

While memory requirements and cost of hardware are obviously significant factors with regard to the use of advanced mathematical software, they are insufficient to explain the lack of impact of the past decade, since programs such as *muMATH* has been available and affordable for this period. Another critical factor, then, lies in the nature of the software itself. As noted above, programs such as *muMATH* may be impressive mathematically, but they fall short when examined pedagogically. If computer algebra is to be accepted into secondary classrooms, then a number of critical features should be present.

The inclusion of a graphics facility has already been mentioned, and the power of the graphical representation in facilitating the learning of much of school mathematics is now well established, both by research and classroom experience (Arnold, 1991c, Demana and Waits, 1988a, Dugdale and Kibbey, 1990, Fey, 1989, Kaput, 1986, 1993, Kissane, 1995, Ruthven, 1990, Tall and Thomas, 1989, Waits and Demana, 1989a, 1989b). Care needs to be taken with the early use of graph plotting software (and the hand-held equivalents) (Demana and Waits, 1988b, Goldenberg, 1988a, 1988b); intensive studies by Goldenberg (1988a, 1988b) and Demana and Waits (1988b) demonstrate the dangers of misinterpretation of graphical information, especially in the early use of such tools. Particular problems have been found associated with inadequate labelling of axis information, leading to poor representation of scale on the part of students. Similarly, interpretation of images viewed through a restricted “viewing window” was found to cause problems among less experienced users. Students who use such tools exclusively may also fail to develop skills and understandings which appear to arise from the physical actions associated with plotting and drawing graphs (Asp, Dowsey and Stacey, 1993a, pp. 53-55).

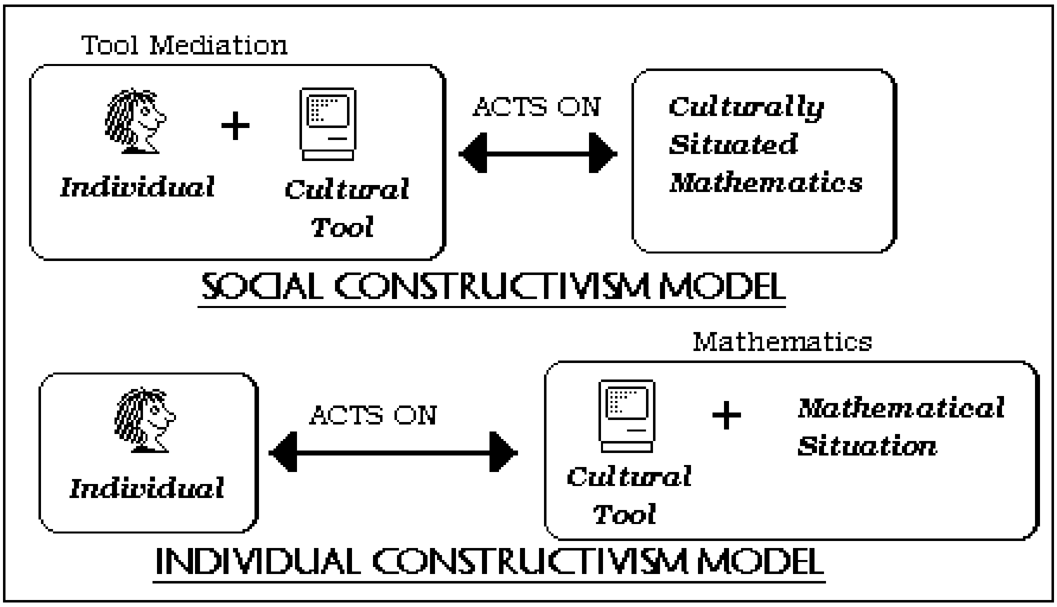
Thoughtful use of the graphical representation of functions, however, has been shown to be a powerful aid to understanding and concept development in secondary and tertiary mathematics (Goldenberg, 1988a, 1988b, Kaput, 1993, Leinhardt, Zaslavsky and Stein, 1990). In an extensive review of research within the domain of functions and graphing, Leinhardt *et al* emphasised the importance of establishing bidirectional links between the graphical and the algebraic forms of functions: “In fact, we might say that this correspondence is *the* thing to be learned” (p. 54). Additionally, the ability of a program to present tables of values for functions may also prove to be of significant advantage in encouraging students to become more versatile in their conceptions (Heid and Kunkle, 1988, Tall and Thomas, 1989, 1991). Such *multiple representation software* has been shown to encourage greater facility with problem solving and functional representations, and to lead to more robust concept development with regard to functions and variables (Afamasaga-Fuata’i, 1992, Arnold, 1992d, Borba, 1993, Harel, 1989, Schoenfeld and Arcavi, 1988, Yerushalmy, 1991b).

Another representational consideration concerns the formatting of both mathematical input and output. It is still common for students to be required to enter mathematical expressions using “computer syntax” rather than true mathematical notation. Even such powerful programs as *Mathematica* and *Maple V* suffer from this failing, although once entered, the input is converted to a reasonable semblance of mathematical formatting. Such programs also use a “command-line” format for instructing the computer: the user must type in the required command in the correct syntax (which usually involves ending with a semi-colon or some other device to signal to the computer that it is time to act). While it may be argued that there is some benefit in encouraging students to enter instructions carefully and correctly, the increased difficulties caused by this approach appear to act as a liability for use in secondary schools - especially if students and teachers are forced to memorise the commands required to operate the program. Programs such as *Theorist, Derive* and *Milo* have minimised this problem through the use of menus and templates, from which the user may select, not only the commands required, but also the notational syntax. There is no need for the user to learn arcane commands or syntactical conventions which vary from program to program, and input and output are both presented in correct mathematical form. Such a feature must be a strong consideration in choosing a program for use in the junior secondary years, and should prove useful with older students as well. In terms of this study, the choice of *Theorist Student Edition* and *Derive* as the preferred software tools was made based primarily upon ease of use and cost factors.

Closely related to the ease of use of available software tools is a sense of personal ownership on the part of the user. Smith (1992) distinguishes between two different senses in which the computer may serve a *mediating* role between the individual and the mathematics which is the object of attention. The first, which he terms the *Social Constructivist Model,* builds upon the Vygotskian notion of cognitive action within a social and cultural context. Within this framework, the image of "computer as intellectual partner" (Smith, 1992, p. 15) is illustrative of a particular mediating role in which child and computer together act upon the mathematics which is “out there”. This culturally situated mathematics also impacts upon the user in a cyclic way (Smith, 1992, p. 18). Distinct from this model is that which Smith designates the *Individual Constructivist Model,* in which the computer, like the mathematical situation, is perceived as being external to the user - the computer is something to be *acted upon*, just as is the mathematics (see Figure 2.1). Both are *external* to the user, whereas within the Social Constructivist framework, the computer is a *personal* tool. Dick (1992) makes a similar connection involving student use of hand-held calculators which, by reason of size and personal ownership are more readily accepted as tools.

This distinction between ways in which the computer is perceived and used by individuals is seen as critical to the current study, and provides a significant area of focus within the research design.

Figure 2.1: Smith’s (1992) distinction between personal and external computer use



Research on Computer Algebra

The earliest empirical study of the effects of the use of computer algebra in the teaching and learning process would appear to be that by Heid (1988), which was based upon her doctoral study completed in 1984. In this quasi-experimental study, Heid compared two groups of college students (n = 39) studying applied calculus, using traditional methods with one group, while the other used graph plotting and symbol manipulation software (*muMATH*) to perform routine manipulations. While the control group spent the full twelve weeks of the course practising the manipulative skills required in the course, the experimental group spent only the last three weeks on skill development. Prior to that, all computations were performed using the computer, while the instructional focus was upon concept development and understanding. Heid found that, while the two groups performed equally well on tests of manipulative ability, the experimental group demonstrated better understanding of the required concepts.

Similar results have been found in other studies comparing groups using computer algebra with those following traditional approaches (Boers, 1990, 1992, Palmiter, 1991). The study by Palmiter used a relatively large group (n = 78) of university calculus students as the sample, and demonstrated not only improved conceptual knowledge as a result of the use of the program *MACSYMA*, but also better computational scores using the computer algebra system than other students using pencil and paper methods (Palmiter, 1991, pp. 153-156).

Boers (1990) investigated the acquisition of the concept of variable in a traditional as opposed to a "computer-intensive" algebra curriculum, in which the program *Derive* was used by a middle-school class of Algebra 1 students to perform all algebraic manipulations. Six students from each group were interviewed twice during the academic year, and the results indicated that the use of computer algebra encouraged in students an understanding of variables as "generalisers" and as "varying quantities that are dependent on other quantities" (Boers, 1992, p. 6), as opposed to the more static concept of variable as an "unknown quantity" which appeared to result from the traditional approach. The experimental group demonstrated also a greater ability to model problem situations and to read and interpret tables of values and graphs than their traditionally instructed peers (Boers, 1992, pp. 4-5). Similar improvements in conceptual understanding without corresponding loss of manipulative facility were supported by other studies involving secondary students using computer algebra systems as scaffolding tools (Rosenberg, 1990, Wood, 1991).

The program *Maple* has been used with both Business Calculus and Calculus 1 students in studies which support the earlier findings by Heid and Palmiter that skill acquisition is not a necessary prerequisite to conceptual understanding or the ability to apply calculus to problem solving situations (Judson, 1990a, 1990b).

These tertiary studies of the use of computer algebra tools differ from secondary school studies in that they involve students of mathematics who, however diverse, must nonetheless be considered select groups of better than average mathematical ability. The secondary studies cited were not concerned with the acquisition of algebraic skills but with the relative understandings associated with particular central concepts. Questions regarding the effects of the use of computer algebra upon skill acquisition among those beginning their studies of algebra remain as yet largely unaddressed by research.

Studies by Yerushalmy have investigated the effects upon junior secondary students of the use of specifically designed tools which combine algebraic manipulation capabilities with graph plotting in what he terms "multiple representation software" (Yerushalmy, 1991a, 1991b). The first study explored the effects of computerised feedback on the ways in which a group of 25 seventh-grade students carried out algebraic transformations, and their approaches to "debugging" their own processes. They were divided into four groups - one which received no feedback regarding their work, another which was simply informed as to whether their results were correct or incorrect; the other groups had computer assistance to provide feedback - one in graphical form, and the other using a symbol manipulator. Students who received feedback on their work in graphical form were found to be motivated to carry out further investigations upon their work - they were able to see the graph of the desired end-product of an algebraic transformation, and the graph of their own result, and were encouraged to act upon their work in order to "correct" their errors. The "manipulator" group was found to be motivated to arrive at the "correct" result using the tools provided. The group given "judgemental" feedback (correct or incorrect) performed better than that with no feedback at all on their end results, but still lacked motivation and probably the ability to correctly complete the tasks. The implications for teaching from this study suggest that computerised feedback in both symbolic and graphical forms can serve as a strongly motivational factor with regard to students performing algebraic transformations, and can encourage them to persevere and continue to explore given tasks to a greater extent than students without such feedback.

Yerushalmy found in another study that software which encouraged students to explore “multiple representations” of functions (symbolic, numerical and graphical) enhanced their understanding of the concept of function and encouraged a more versatile approach to problem solving. This study, with 35 eighth graders, also suggested, however, that the cognitive “links” between the various representations were fragile, and did not occur spontaneously (Yerushalmy, 1991b, pp. 54-55). Student misinterpretations and informal theories were common, as was the tendency to view functions as objects rather than processes. There is a need to implement the use of such powerful computer tools thoughtfully and with caution (Yerushalmy, 1991b):

> During this course the students developed and continued to use theories which reflected both their correct understandings and their misconceptions. As Goldenberg (1987) claims, such theories might not be efficient, but they have their own value. As it was shown, these theories do not impair student ability to carry out correct techniques in an efficient way. (p. 55)

There exists, then, a growing body of evidence which suggests the benefits of the use of computer algebra software as an aid to mathematics learning and instruction in secondary schools. Used carefully, such tools may be expected to contribute to improvements in concept development and understanding, in addition to more positive attitudes towards the subject. Students using such aids have been found to be more effective and persistent problem solvers, and more inventive in their approaches. No evidence has been found to support claims of reduced manipulative ability when such software is used to support computation.

The use of computer algebra, however, is not without its critics. In particular, Waits and Demana (1988) argue that little is to be gained from “a device that simply symbolically produces the answer” (Waits and Demana, 1988, p. 334). They argue the case strongly for the use of graph plotters and particularly graphics calculators as essential aids to mathematics instruction. In dismissing the use of computer algebra, however, they suggest the following as reasons (after Waits and Demana, 1992):

- Cost and accessibility preclude the majority of students from using such technology, both at home and school;
- ... (E)xact answers produced by computer symbol manipulators are often of no real use and sometimes furnish little insight into the problem modelled by the algebraic representations.
- ...(S)tandardised tests change slowly and students will be required to demonstrate some ability in algebraic manipulation on them for some time to come...
- ... (N)o one can be sure at this time how much paper-and-pencil algebraic manipulation is really necessary for success in college and in a work place that requires increasing technological and scientific know-how. (p. 180)

While such arguments provide valuable cautionary guidelines from which to view the use of computer algebra software in schools, they fail to adequately argue against their introduction. Many of the same criticisms regarding access and affordability have been levelled against graphics calculators in the past, prior to their reductions in price in the past two years. The same reductions and increased availability may be expected to apply to computer algebra tools, with “free” and “shareware” programs such as *MathMaster* and *CoCoA* for the Macintosh, and *CC3 - the Calculus Calculator* for MS-DOS machines already available. The problem of student access to hardware, of course, remains critical.

The issue of *personal access* to classroom computer technology for the students (discussed above) may be directly related to the development of hand-held mathematical computers which offer graph plotting and, in some cases, symbol manipulation (Ruthven, 1992, Dick, 1992). “Portable” computer algebra systems such as the Hewlett Packard HP-28 series have been available for some years, and present another alternative to the problems of access and equity which have already been mentioned. Although expensive in comparison with numerical calculators and even graph plotters, the cost of a class set of such tools (eight to ten units, allowing senior students to work in groups of two or three) is comparable to that of a classroom computer. Research on the use of such tools in senior classes suggests that they can contribute to improvements in concept development and understanding, student attitudes and confidence (Arnold 1990, 1992e). The symbolic manipulation capabilities of these calculators are, at present, limited - they tend to be slow and difficult for students to master, and so do not provide a sufficiently transparent and intuitive tool for use across the secondary years. The advantages of portability and personal access, however, make their use in the senior years attractive.

The argument that exact answers to symbolic problems actually obscure interpretation of the solution (Waits and Demana, 1992, p. 181) is an interesting one; however, most computer algebra tools allow answers to be expressed in both exact and rational approximation forms, a choice not available on hand-held computers. Similarly, criticising computer algebra systems because they sometimes produce different forms for the same answer seems also to be somewhat short-sighted, since this provides a powerful incentive and opportunity for student exploration of mathematical identities, and certainly a valuable basis for classroom discussion.

In summary, then, the literature provides adequate evidence of the potential for computer algebra tools to enhance the teaching and learning of mathematics at all levels from the junior years of secondary school to the tertiary years. While the current state of the research in this area remains far from conclusive, and there is evidence that there are dangers and pitfalls in the use of such technology, there appears to be much to recommend further research on the effects and implications of such instructional practice. In the absence of extensive research evidence with regard to this new and expanding educational field, the value of further research is heightened. As increasing numbers of educators begin to explore the classroom implications of advanced mathematical software, their observations and opinions provide valuable guidelines for others entering the same largely uncharted waters.

The Contribution of Vygotsky

Both Piaget and Vygotsky began their studies in the early years of this century. While Piaget's life and work spanned most of this century, Vygotsky died in 1934, at only 38 years of age. His work was largely unknown outside his own country until the 1960s; even within the Soviet Union, it was suppressed for many years. Over the past three decades, however, Vygotsky's writings have assumed growing importance, particularly in the development of wholistic theories of language learning, but increasingly in fields such as mathematics learning and teaching (Zepp, 1989, Manning and Payne, 1993, Confrey, 1993b). His notion of a *Zone of Proximal Development* has proved appealing in a wide variety of contexts, and offers much in the present exploration of the way in which appropriate computer tools may assist the growth of understanding.

In seeking to understand something of the contribution of Vygotsky in the present context, it is appropriate to begin with the notion of "tools". He began his work, *Thought and Language* (Vygotsky, 1962) with a quotation (in Latin) from Sir Francis Bacon, translated by Bruner (1986) as:

> Neither hand nor mind alone , left to itself, would amount to much. And what are these prosthetic devices that perfect them? (p. 72)

The additional tools to which Vygotsky appears to be referring are, most importantly, thought and language, those means by which we are recognised as most uniquely human. In opposition to Piaget, Vygotsky places language as the precursor to thought, claiming that it is only through the use of language that the higher mental processes may develop and become operational. Language is social in origin, developed through interaction with others, and, in Vygotsky's view, serves two primary purposes - self-direction and communication. This perception of language as a tool which aids thought is a fundamental feature of

Vygotsky's view. He believed that the higher mental processes are *mediated* by language, first observed as egocentric speech in children, which then becomes internalised, developing into thought. (Zepp, 1989, p. 30-32)

Mathematics shares with language these twin characteristics: it is, at once, both cognitive tool and means of communication. This distinction (drawn by Confrey, 1993b, p. 50) is significant in the context of the use of computer technology in mathematics learning. In its role as tool, mathematics may be perceived primarily as a means of effecting some outcome; as Confrey points out, the image of mathematics as *tool* links it with *action*, a significant aspect often overlooked. The potential for computer technology to assume an active mediating role in supporting mathematical thinking, learning and practise, is critically important in the current study. The potential role of goal-directed action may be exemplified by comparing the interface offered by the *Macintosh* computer algebra package, *Theorist*, with that of other packages of the same type. Most computer algebra tools allow the immediate solution of equations, for example, through a general "Solve" command; in some cases, the intermediate steps of the equation-solving process are supported by allowing operations to be carried out on both sides of the equation. *Theorist* is unique in allowing the user to physically manipulate the terms and elements of algebraic expressions and equations. Solving an equation such as

$$\frac{3}{x - 1} = 2$$

may be achieved by physically selecting the denominator, **x - 1**, and, using the mouse, dragging it across to the right-hand side of the equation, to automatically produce

$$3 = 2 (x - 1)$$

This may be expanded, and the **x** term isolated by similar manipulations. The program offers the choice of alternative techniques which, in some ways, may be preferable pedagogically (involving performing the same operation to both sides of the equation, producing equivalent equations). At the same time, many students and teachers solve equations in exactly this way, involving either overt or covert physical manipulation of the terms. The role of action in higher mathematical processes is likely to be significant, but as yet remains largely unexplored. This recent development of computer software which simulates and supports such physical involvement invites such exploration.

In the context of the present study, the thinking of individuals as they interact with advanced mathematical software is seen to be accessible through the twin avenues of *action* and *language*. The software tool designed for the study captures both aspects of the interactive process. It records the actions of the users as they engage in a wide range of mathematical tasks, both independently and within the context of the instructional modules provided - each button that is pressed, each option chosen, the time spent at each point of the process; these important elements of physical involvement become part of the record of interaction which the software provides. Written comments in response to questions, prompts and probes generated by the software throughout the session also become part of this record, allowing the language of the user to be coupled with the concurrent interactions.

Central to Vygotsky's view of cognition and learning is the social and cultural context of the learner. In particular, the learner is viewed as achieving higher cognitive ground through interaction with others, especially adults and knowledgeable peers. His *zone of proximal development* may be thought of as “the distance between the actual developmental level as determined through independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers” (Vygotsky, 1978, p. 86). Bruner explores the implications of this concept for tutoring and guided learning through his notion of “scaffolding” (Bruner, 1986, pp. 74-76). Since the use of the software by students in the present study occurs most often within such a context, these ideas are particularly significant.

Wood (1980, 1986) expands upon both Vygotsky’s zone of proximal development and Bruner’s notion of scaffolding to offer a model of learning based upon twin central principles of *uncertainty* and *contingency*. Wood observes that learning within a situation of uncertainty is always less effective than one in which the learner is able to recognise commonalities and familiar features. Motivation, task orientation, even the ability to remember particular features of the situation - all are likely to be reduced in unfamiliar situations, and much mathematics learning occurs within situations high in uncertainty. Support is required, then, in such situations of high uncertainty which will serve to alleviate these problems and so make the learning experience more effective.

Wood’s second key principle defines the preferred nature of such support, requiring that the response of the tutor be *contingent* upon that of the child if optimal cognitive progression is to occur. Wood postulates five levels of increasing control which may be observed in a tutorial situation (that is, a learning situation involving interaction between the learner and a more capable other - the “tutor”) (Wood, 1986, pp. 197-198). These range from minimal control - the tutor prompts the learner with a general question, such as “What might be done here?” - to highly controlled, in which the tutor actually demonstrates the steps needed to fulfil the requirements of the task. Wood’s principle of contingency requires the tutor to decrease the level of control at each correct action of the learner, and to increase the level of control or intervention upon each error. This process of flexible scaffolding allows the learner to progress optimally across the zone of proximal development, with greater and lesser degrees of support as required. The critical principle in such learning is the promotion of autonomy and independence on the part of the learner. It is not difficult to support and scaffold learning; the challenge lies in doing this in such a way that the scaffolding is gradually removed, and the learner actually decreases the level of dependence upon the support structure as the learning sequence progresses. This is the primary goal of contingent learning. In the context of mother-child instructional situations, Wood cites research which supports such learning: “What we find is that the more frequently contingent a teacher is the more the child can do alone after instruction" (Wood, 1986, p. 198).

A common assumption has been that the necessary support must be given by others; the potential role of the computer in such a relationship remains open. In particular, the use of software which supports the manipulations and representations of high school algebra appears to coincide with this notion of scaffolding. Certainly, if used to complement an adult-child tutoring relationship all requirements are fulfilled. At the same time, two other scenarios may be considered: the first involves two peers, both beginning their study of some aspect of algebra, with access to computer algebra software. Although neither may serve the role of “expert”, their social interaction and verbalisation while working together using the computer as support (effectively assuming the “expert” role) appears to offer much in common with the Vygotskian notion. Certainly, the situation offers the opportunity for the verbal and social interaction so necessary for the achievement of cognitive progression, while the software offers the means of navigating across the zone of proximal development, allowing work in advance of the current state of both students.

Another scenario, too, may be considered. In this situation, the role of “expert” is again taken by the computer, but this time in the form of interactive instructional material prepared in advance. The individual student may work through such materials supported by access to suitable advanced mathematical software which will aid and encourage enquiry and exploration. The student enters responses and comments, and answers questions as they occur in the work, forcing verbalisation of the developing ideas. This model, too, shares much in common with Vygotsky’s model of learning, in which the learner is challenged to move beyond the present level, and is supported in this movement by access to appropriate software. The *Exploring Algebra* package (Arnold, 1993) has been developed using *HyperCard* on the *Macintosh* computer to provide just such a learning model, especially when used in conjunction with an appropriate support program, such as *Theorist*

Student Edition.

The structure of the learning environment is the focus for research by Valsiner (in Rogoff and Wertsch, 1984) which extends the study of the zone of proximal development. Exemplified by the adult-child learning experiences associated with the socialization of meals, Valsiner proposes two additional zones which serve to define more clearly some of the situational constraints which may act to support or impede progress across the zone of proximal development. The first of these constructs, called the *zone of free movement* (or ZFM) is based upon the observation that learning is facilitated by focussing the attention of the learner upon that which is to be achieved. This may be done by restricting the actions of the learner, or by defining a “zone of free movement” (Valsiner, 1984).

> Within the field of objects and affordances related to them in the environment of the child, the zone of free movement (ZFM) is defined for the child’s activities. The ZFM structures the child’s access to different areas of the environment, to different objects within these areas, and to different ways of acting upon these objects. (pp. 67-68)

In the present study, this zone is defined by the boundaries of the software, with the *HyperCard* modules serving as the base from which other tools may be readily accessed, and then returning once again to continue with the task at hand. The parameters of the ZFM, then, are clearly defined within the context of the computer tools available.

Defined conjointly with the zone of free movement is a *zone of promoted action* (ZPA). If the ZFM is effectively an “inhibitory mechanism” (Valsiner, 1984, p. 68) which functions to limit the actions of the learner within the structured environment, then within that zone exist "sub-zones" which are defined by those actions sought to be encouraged and learned. In the context of "meal time", these may involve the appropriate use of cutlery; in relation to the present study, the zone of promoted action will be defined by the appropriate use of available software tools to achieve mathematical goals. In particular, the ready accessibility of computer algebra, graph plotting and table of values utilities encourage their use by the learners; the extent and form of such use becomes the primary focus of this study.

Theory based upon the work of Vygotsky, then, offers much which may inform and direct a study of the use of advanced mathematical software. Such theories provide significant guidance in the search for ways in which such software may be used, and such use studied. There is a need, however, for more descriptive detail if the interactions of students and teachers with the technology are to be made explicit. In particular, Vygotsky recognises the existence of qualitatively different levels of development and styles of thought, but does not pursue or expand on these. If the algebraic thinking and understanding, and the thinking about teaching and learning, of both students and teachers are to be observed and monitored, then such detail is a necessity. In the domain of mathematical thinking, the theory of Pierre van Hiele and Dina van Hiele-Geldof offers a suitable framework.

A Theory of Mathematics Education

Although van Hiele's theory (van Hiele, 1986) has been most widely recognised for its role in explicating the levels of thinking associated with the learning of geometry, it has been developed as a general theory of mathematics education. Growing from the concerns of teachers, it does not stop at the description of "levels of thinking", but seeks to provide a basis for understanding the movement between these levels, and the role of the teacher in assisting such progression. As such, the theory goes beyond the SOLO taxonomy (which is essentially descriptive) and beyond, too, the concerns of Piaget, who deliberately distanced himself from the question of how students may be encouraged to progress from level to level. His was a developmental theory, holding that such progression was largely independent of the influence of instruction; he referred to such concerns disparagingly as "the American question", but in fact it was the Dutch van Hieles who appear to have made significant progress in addressing it.

In his recent work (1986), Pierre van Hiele describes a theory of mathematics education arising from the study of two fundamental concepts - structure and insight. Although reluctant to specify a definition for the first, van Hiele admits that it may be broadly thought of as a "network of relations" (van Hiele, 1986, p. 49) in which commonalities are recognised across all types of events and perceptions. In everyday life, we recognise structure in going through daily routines, at work and home; structures are apparent in the patterns of nature and man; continuing a sequence of numbers is a recognition of structure, as is the recognition that the symbol $\mathbf{(x + 2)^2}$ may be seen as a sign to expand the given expression and produce a new equivalent one. *Insight*, in this context, is a recognition of *structure* - we know what to do when we experience such insight, and it is precisely an absence of such insight which leaves so many school students at a loss as to know what to do with a given algebraic expression, equation or problem.

Van Hiele distinguishes between *rigid* and *feeble* structures (van Hiele, 1986, pp. 19-23), strongly reminiscent of Wood's principle of *uncertainty* (Wood, 1986). Consider, for example, a student presented with the expression, $\mathbf{x^2 - (x + h)^2}$. The more likely response for a student of at least moderate algebraic facility is to attempt to expand and simplify the expression. The recognition of the requirement to expand the squared part of the expression may be thought of as a relatively rigid structure. The recognition that such an expression provides an opportunity for factorisation, as a "difference of two squares", however, is likely in most students, to be a relatively feeble structure. Some prompting may be required for students to recognise this structure, even when they quickly recognise it in a case such as $\mathbf{x^2 - 4}$. The dominant strategy of algebra instruction in the past has centred around the development of rigid structures through repetition, seeking to "automate" student responses to algebraic prompts. Such learning, however, is likely to occur at a very superficial level, and is relatively easily exposed when students encounter an exceptional case. As explained by Confrey, this relates closely to the Vygotskian notion of "pseudoconcept" (Confrey, 1993b),

> ... acknowledging that children often use words before they have grounded its [sic] meaning in conceptual operations. Vygotsky suggests that this use of language that runs ahead of cognitive depth is an important part of learning - and describes a key mechanism in how adults teach children to advance to higher levels of cognitive thought. (p. 50)

This recognition of the central role of language in the learning process is a common theme throughout the works of both Vygotsky and van Hiele.

To van Hiele, true learning is that which students achieve through their own efforts, efforts which involve them in experiencing what he terms a "crisis of thinking" (van Hiele, 1986, p. 43). Similar to the Piagetian notion of disequilibrium, and very close to Doll's "perturbation" (Doll, 1986, p. 15), van Hiele sees such a crisis as necessary for students to achieve a higher level of thinking. While teachers may be successful in having students "mimic" the responses of a higher level, unless the learner has struggled with the material personally, no cognitive gain has been made. The cognitive "safety nets" (described by Tobin and Fraser, 1988) which are a feature of many mathematics classrooms are attempts by students (and by their teachers) to reduce the cognitive load of the material to be learnt; such efforts in van Hiele's view, must be carefully controlled, since meaningful learning involves transition to a higher level of thinking, and this can only occur by going beyond the present state. The links with Vygotsky's zone of proximal development are apparent, where "the only good learning is that which is in advance of development" (Vygotsky, 1987, p. 89).

This is the point at which the theories of learning described here coincide. For all their various forms and distinct priorities, the common ground is the perceived need for *challenge*. The teacher does not encourage learning by predigesting the material; rather, the learner must be an active participant in the process of creating meaning through interacting with that which is to be learnt in a context which supports exploration, verbalisation and activity.

The ways in which the *levels of thinking* proposed in the van Hiele theory (van Hiele, 1986, p. 53) complement those of the SOLO taxonomy have been described in detail elsewhere (Pegg, 1992a). The van Hiele levels begin, not with the level of *action* proposed as the sensori-motor mode of the SOLO taxonomy, but with the level of *visualisation* or *recognition* (Hoffer, 1981), corresponding to the global, intuitive thinking associated with *ikonic* thought. Next is the level of analysis, or the *descriptive level* (van Hiele, 1986, p. 53), corresponding closely to the concrete-symbolic mode of the SOLO taxonomy. This is followed by a level alternatively labelled *abstraction* (Burger and Shaughnessy, 1986), *ordering* (Hoffer, 1981, p. 14) or, simply, the *theoretical level* (van Hiele, 1986), easily recognised as encompassing *formal* modes of thought. Although the literature describes successive levels (commonly as *deduction* and *rigour*) van Hiele himself appears more inclined to view these as logical extensions of the *theoretical level* (van Hiele, 1986, p. 53) which, once achieved, experience a phenomenon he describes as *level reduction* (van Hiele, 1986, p. 53). Although the objects of thought may involve successively higher levels of abstraction, the actual mode or style of thinking remains essentially the same. The correspondences which occur with the SOLO taxonomy enable the two models to be considered as logically compatible; the different ways in which each illuminates each style of thought, however, makes the synthesis proposed here attractive.

The theory of van Hiele, in addition to describing levels of thinking, offers an important addition. This is the notion of *stages of learning* as means by which the learner may be assisted to seek higher cognitive ground. Five such stages are specified (van Hiele, 1986):

1. In the first stage, that of *information,* pupils get acquainted with the working domain.

2. In the second stage, that of *guided orientation,* they are guided by tasks (given by the teacher, or made by themselves) with different relations of the network to be formed.

3. In the third stage, that of *explicitation,* they become conscious of the relations, they try to express them in words, they learn the technical language of the subject matter.

4. In the fourth stage, that of *free orientation,* they learn by general tasks to find their own way in the network of relations.

5. In the fifth stage, that of *integration*, they build an overview of all they have learned of the subject, of the newly formed network of relations now at their disposal. (pp. 53-54)

These stages of learning are significant in providing a framework for instruction aimed to develop understanding of the material or skills to be learnt. Each of the five stages relates to an aspect of the *HyperCard* program, *Exploring Algebra* (Arnold, 1993) developed for this study of the ways in which teachers and students interact with advanced mathematical tools. The program was designed to provide information about the topic to be studied, using the point and click interface of the *Macintosh* and the branching features offered by *HyperCard* (Stage 1). Problems are posed within the materials (Stage 2), and computer tools made easily accessible by which such problems might be investigated (Stage 4). Although a “comment” option is provided, it remains for the teacher or tutor to pursue the verbalisation required for stage 3, and to draw together the materials for the last stage of integration. Probes and prompts (described below) seek to elicit responses concerning understanding of the material and thinking about the critical concepts of algebra (such as equations, functions and expressions), but the student should ideally share their thinking with another at some stage in the process. This illustrates a further consistency with the Vygotskian notion of the zone of proximal development.

The SOLO Taxonomy

Both pedagogical and mathematical thinking may be viewed as consisting of a range of elements, operating on different levels. The SOLO taxonomy provides valuable insights into the nature of such elements. Building upon the developmental learning theories of Piaget and Bruner, Biggs and Collis recognised that learners demonstrate distinct modes of functioning, generally corresponding to the following age periods:

From Birth .... *Sensori-Motor*
From around 18 months. *Ikonic*
From around 6 years ...... *Concrete-symbolic* (approx. K-Year 10)
From around 16 years.... *Formal* (approx. Years 11 and 12 +)
From around 20 years.... *Post-Formal* (University/professional practice)

Differing from classical stage theory, it is not suggested that each stage *replaces* the previous one, but that each adds to the available cognitive repertoire. In different situations, learners may "regress" to an earlier acquired mode of functioning or utilise a higher cognitive function in the learning of a lower-order one, adopting a "multi-modal" approach to the task at hand (Biggs and Collis, 1991, Collis and Biggs, 1991). An example of the first situation (labelled "top-down" learning by the authors) would be the use of intuitive, visual methods in mathematical problem solving, where the *ikonic* mode is used to supplement the more usual *concrete-symbolic* approach. "Bottom-Up" learning may be illustrated by the use of higher-order practices in the learning of sensori-motor skills (such as thinking through the action of a golf swing, studying the style of expert players or learning the theory and techniques of art in order to improve in the ikonic aspects). Although much of secondary schooling may be recognised as occurring within the *concrete-symbolic* mode, the use of multi-modal strategies may be more extensive than previously realised (Biggs and Collis, 1991, Collis and Biggs, 1991). It is certainly common in areas such as music, which utilises sensori-motor, ikonic and concrete-symbolic elements in the learning process, and recent research suggests that the ikonic mode may be a powerful influence in mathematical problem solving (Collis, Watson and Campbell, 1992). At present, however, much of the focus of instruction in secondary schools lies within the *concrete-symbolic* domain. Even at the senior level, it is now believed that the end-point of instruction in most subjects will be at this level. Only in those areas in which the student is particularly competent (and likely to continue into tertiary study) is *formal* mode functioning likely to be observed with any degree of frequency (Collis and Biggs, 1992).

Some learners never reach the *formal* stage, at which the foci of interaction are theories and abstractions, rather than the more concrete objects of earlier stages; many, perhaps most, do not achieve *post-formal*, which involves working with and extending theory *systems* themselves. With increased retention rates in the senior years of schooling, it is likely that increasing numbers of senior students will be operating throughout their studies at the *concrete-symbolic* level (Collis and Biggs, 1983). The preferred mode of operation for students has significant implications for learning and instruction (Collis and Biggs, 1991), and will become a focus for this investigation in the study of the representation and understanding by students and preservice teachers of algebra and learning interactions.

Further, Biggs and Collis suggest that, *within* each mode of functioning, learners display a consistent sequence or “learning cycle” when learning new tasks. This gives rise to the SOLO acronym, detailing the “Structure of the Observed Learning Outcomes”. The theory postulates five distinct levels or outcome structures (from Biggs and Collis, 1989):

Prestructural	The task is engaged, but the learner is distracted or misled by an irrelevant aspect belonging to a previous stage or mode.
Unistructural	The learner focuses on the relevant domain, and picks one aspect to work with.
Multistructural	The learner picks up more and more relevant or correct features, but does not integrate them.
Relational	The learner now integrates the parts with each other, so that the whole has a coherent structure and meaning.
Extended Abstract	The learner now generalises the structure to take in new and more abstract features, representing a higher mode of operation. (p. 152)

Using this framework it becomes possible to identify an individual’s current level of operation for a particular task through a study of verbal and/or written responses. It thus provides a powerful tool for the assessment of student understanding of concepts, and for problem solving (Collis and Romberg, 1991). The taxonomy has also proved effective as a means of planning and developing curricula based on the cognitive characteristics of the learners (Stanbridge, 1990).

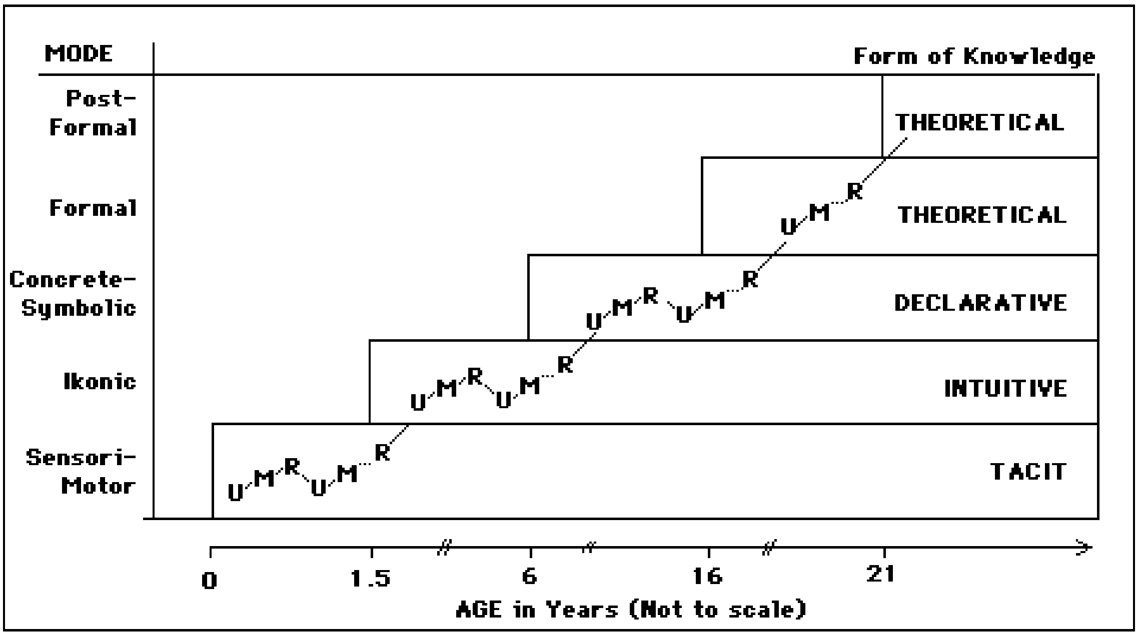
In terms of pedagogy, the unistructural, multistructural and relational levels are recognised as the “target modes” for teaching; allowing for individual differences, it may be expected that all students should achieve one of these levels as a result of an effective learning experience. In the case of new work, it should be the teacher’s objective to assist the students to move from a prestructural state (with no organised or coherent knowledge of the material) to one that is, ideally, relational. In practice, however, the more likely end result for instruction is *multistructural*, in which students and their teachers are satisfied to know “some things about an area”. Relational understanding (in both the SOLO sense and that of Skemp, 1976) is frequently sacrificed for the demands of utility. The occurrence of an extended abstract response is not normally one that is anticipated as a direct result of instruction, but is more a function of the individual learner’s ability to go beyond what has been taught.

Recent research suggests that, just as the linear sequence of modal development originally proposed has given way to a more complex multi-modal structure, so the cycle of levels within the modes may be more complex than originally anticipated. In particular, at least two Unistructural-Multistructural-Relational cycles now appear to exist within the *concrete-symbolic* mode, as observed across a range of mathematical topics in the junior years of high school (Pegg, 1992b).

This would help to explain quite distinct styles of thinking about complex mathematical objects (such as process and object conceptions of functions) while situated within a single cognitive domain. As more is revealed through research, the model of cognitive development offered by the SOLO Taxonomy assumes less of the linear, sequential pattern of its Piagetian origins, and more of a complex branching structure.

Figure 2.2 (Pegg, 1992a, p. 27) provides a schematic outline which relates modes, learning cycle, curriculum goals and suggested exit levels for schooling. Each mode of operation is associated with a particular “type of knowledge”, as illustrated. That arising from the *sensori-motor* is likely to be *tacit*, unable to be articulated, as in the “feel” of a good golf swing. The *ikonic* mode produces knowledge which is *intuitive*, difficult to verbalise, and closely linked to visual and emotive aspects of the situation. The *concrete-symbolic* mode leads to knowledge that is *declarative* - not only knowing “how to”, as in earlier modes, but being able to say “why”, at least in terms of the concrete referents available. The *theoretical* knowledge which results from the later modes involves adopting theories and theory systems - complex networks of relations between ideas and concepts - as the objects of thinking.

Figure 2.2: SOLO Taxonomy: Schematic Outline



Images and Definitions of Algebra

The concept of "function" is increasingly recognised as central to an understanding of algebra across the years of secondary schooling and beyond, particularly within a technology-rich context (Harel and Dubinsky 1992, Grouws, 1992, Romberg, Fennema and Carpenter, 1993). Senior students are expected to be familiar with a range of common functions, including the linear, quadratic, trigonometric, exponential and logarithmic functions; to sketch, manipulate, differentiate and integrate them. The study of functions occupies the major part of the time spent on mathematics in the senior school.

The concept of "function", not surprisingly, is one which is mathematically rich, capable of being thought of using a number of distinct representations, or "images". Numerous studies over the past decade have investigated the ways in which different groups of people think about and use functions. Barnes (1988, p. 121), in interviews with secondary and tertiary students in New South Wales, identified six frequently occurring images of functions:

- A graph or curve
- A set of ordered pairs or table of values
- A relationship between two variables
- An algebraic formula or equation
- A "function machine" (input-output device)
- The symbol f(x)

She also distinguished the image of function as a "*mapping between two sets*", which was not widely used by students, but was seen as useful in thinking about the concept.

Barnes interviewed what she described as a "small group of Year 11 students, who had recently begun calculus" (Barnes, 1988, p. 119). She found that the dominant image of function used by these students was a *graphical* one and that there was widespread uncertainty about "what a function is" (Barnes, 1988, p. 122) - presumably the more formal definition of the concept. A more recent and extensive study, which was undertaken to provide baseline data for the current project, found a different pattern of representation (Arnold, 1992d). This study, of close to 400 high ability secondary and tertiary mathematics students found that these students were most likely to think of functions as algebraic formulae or equations, and slightly less likely to use a graphical image, or to think of them as rules or relationships between variables.

Studies by Vinner and his colleagues over the past ten years (Vinner, 1983, Tall and Vinner, 1989, Vinner and Dreyfuss, 1989) distinguish between a "concept image" ("the set of all the mental pictures associated . . with the concept name, together with all the properties characterising them" (Vinner and Dreyfuss, 1989, p. 356)) and a "concept definition" ("a verbal definition that accurately explains the concept in a non-circular way" (Vinner, 1983, p. 293)). Such studies reveal, among other things, that concept images may not always be consistent with the formal definition, but that such inconsistencies are often not apparent. Much of the focus in this area has been upon identifying individual images which students prefer to use when thinking about functions, although the verbal descriptions given as definitions of functions frequently comprise multiple images. This was further supported in Arnold (1992d), in which the pattern of representation was quite different when students were asked to describe a function "in their own words". In this situation, functions were most likely to be described as a "rule or relationship", which coincides with the common (non-mathematical) idea of "function", or one of several "multiple-image" definitions, such as an "algebraic object which can be graphed", or a "rule which can be expressed algebraically or graphically" (Arnold, 1992d).

The SOLO taxonomy distinguishes levels of understanding in the acquisition of concepts which are relevant in this context. Students who focus quickly upon a single property or characteristic of a concept are said to be *unistructural*; those who recognise several properties as relevant, but do not link these together may be thought of as *multistructural;* those who are able to see the relationships between the various properties or representations of a concept are said to be *relational*. Some may go beyond this level, forming new connections and seeing applications of the concept in new situations; such learners are said to be operating at an *extended abstract* level. These levels (together with an initial *pre-structural* level) are considered to cycle through each of the developmental modes - sensori-motor, ikonic, concrete-symbolic, formal and post-formal operations. Learners who describe functions using multiple images would be considered to be operating at a higher cognitive level than those who use only a single image. Whether they see the relationships between these images (relational) or merely perceive them independently (multistructural) would be difficult to determine without individual interviews. Nonetheless, we might expect that those who describe functions using multiple representations might be more successful than those who think of them unistructurally in solving problems which require analysis of function properties. More recent developments in SOLO theory (Biggs and Collis, 1991, Collis and Biggs, 1991) suggest that individuals who operate in a *multimodal* way (able to draw upon earlier modes of thinking) will be more effective as problem solvers and critical thinkers. Students able to draw upon versatile images of function (including the idea of “function as process” and global “ikonic” images) to supplement their more usual concrete-symbolic way of thinking were found to be more capable at analysing functions and solving problems than those who tended to approach such situations unimodally (Arnold, 1992d).

The mental representations of functions have been described in a variety of ways. Eisenberg and Dreyfuss (1989) distinguish between *visual* and *symbolic* representations, similar to that which Vinner (1989) describes as *visual* and *algebraic* modes of thinking about functions.

Konvisser (1989) describes three representations - numerical, symbolic and graphical, while Tall and Thomas argue for *versatile learners* (1989) - citing the work of Brumby (1982), they describe people as global/holistic, exemplified by thinking of functions as curves, serialist/analytic, as in the “function machine” image or, more commonly, “versatile” - a mixture of the two.

A dual view of mathematical concepts such as function is further explored by Sfard (1991, 1992, 1994), who describes such concepts as having *structural* and *operational* dimensions, which should be seen as complementary, not incompatible. Viewing functions as objects (whether algebraic or graphical) is consistent with a structural conception, while the idea of function as process is an operational view. Sfard argues that the two perspectives interact and support each other, and that the operational mode precedes the structural mode in concept formation. In learning a concept such as function, students begin with an active conception (substituting numerical values into expressions) and gradually come to view such expressions as objects in their own right. Such objects may then be manipulated and analysed for their own sake, eventually becoming the basis for new processes in the formation of other higher order concepts (as in the composition of functions). This process is consistent with that described by Bruner (in Bruner and Anglin, 1973), who sees cognitive growth as moving through stages of representation which he described as *enactive*, *iconic* and *symbolic*.

The evolution of the function concept throughout secondary schooling follows this pattern. Although students begin with the enactive ideas of "function machine" or numerical substitution in junior secondary, they move quickly to the study of functions as objects, which appears to remain the focus for all future study. The valuable perception of function as process may well be lost for many students by the time they reach their senior years. Whether they are fixed at the analytic concrete-symbolic mode or the global ikonic mode, early research indicates that students unable to utilise both ways of thinking may well be disadvantaged in thinking about and using functions effectively (Arnold, 1992d). More recent work by Sfard (1992) suggests that the majority of secondary school students hold a concept of function which she terms *pseudostructural*, a superficial and inflexible understanding which results from teaching the concept as an object (that is, teaching *structurally*) before the students have established its nature and reality through exploration of the *operational* dimension (Sfard, 1992, pp. 75-77).

As the use of computer technology has begun to impact more and more upon senior mathematics classrooms in the form of *graph plotting* software, *multiple representation software* (such as *ANUgraph* and *CC3 - the Calculus Calculator*) and *computer algebra tools* (such as *Maple V*) which allow both the representation and manipulation of functions in a variety of forms, the effects of such technology upon the ways in which students visualise and use mathematical functions become critically important. The use of such technology to improve student understanding of the concepts of function and variable, and to strengthen the links between the various representations available for such concepts is likely to become a significant factor in the effective use of computer tools in mathematics learning, and provides a central focus for the present study.

Effective Teaching in the Computer Age

In order to provide initial direction regarding the implementation of the new modes of instruction which will accompany the use of computer algebra tools in mathematics classrooms, it is relevant to consider previous research on effective teaching in a more general sense. In particular, does the extensive research on teaching reveal particular instructional behaviours, teaching strategies or modes of thinking and representation which are likely to aid in the successful implementation of the current technological innovation? Studies on the classroom use of computers in a variety of contexts indicate considerable promise as demonstration tools (Ganguli, 1992), especially with regard to improved concept development, and for encouraging a more individualised instructional mode (Hativa *et al*, 1990). Work with pre-schoolers suggests that young children interact naturally and effectively with computers without the level of adult intervention previously considered necessary (Hall and Elliott, 1992). However, there is also evidence that teachers are unlikely to alter their instructional patterns to any real extent in order to incorporate classroom computer technology - even to the extent that they are unlikely to rearrange pre-existing spatial arrangements or alter the sequence or mode of instruction (Mehan, 1989). There was evidence, however, that students were more adaptable in such circumstances, and showed increased levels of mutual assistance and co-operation. The problems of incorporating computer technology into effective instruction, then, may well lie far more with teachers than with the students involved.

What constitutes "effective" or even "good" teaching, of course, remains problematic, particularly in a time of "paradigm shift" (Prawat, 1992, p. 354) from previous assumptions about teaching and learning to the constructivist stance increasingly espoused by researchers, if not by practitioners (Prawat, 1992, Richardson, 1990). As new priorities for instructional outcomes are recognised, practice previously recognised as "effective" may suddenly become inadequate. A study by Schoenfeld of a teacher in a 10th-grade geometry class demonstrated exactly this effect (Schoenfeld, 1988):

> Two pictures of the instruction and its results emerged from the study. On the one hand, almost everything that took place in the classroom went as intended - both in terms of the curriculum and in terms of the quality of the instruction. The class was well managed and well taught, and the students did well on standard performance measures. Seen from this perspective, the class was quite successful. Yet from another perspective, the class was an important and illustrative failure. There were significant ways in which . . having taken the course may have done the students as much harm as good. (p. 145)

Attempts to classify teaching behaviours as "exemplary", "expert" or even "effective" must be viewed as a consequence of the assumptions about teaching and learning which one holds. In particular, the method of assessment which is used will have considerable influence upon the conclusions drawn about a particular situation. If the outcome of teaching practice is measured only in terms of student achievement on traditional examinations, then certain strategies may be considered well-defined in contributing to such results. Correlating teaching practices with class achievement in introductory algebra, LeClerc, Bertrand and Dufour concluded in agreement with Good Biddle and Brophy (1983 in LeClerc, Bertrand and Dufour, 1986),

> that pupils learn more efficiently when their teachers first structure new

> information for them and help them to relate it to what they already know, and then monitor the performance and provide corrective feedback during recitation, drill, practice, or application activities that provide pupils with opportunities to develop mastery and use what they have learned. (p. 365)

If such advice is offered as a "recipe" for "effective teaching" then, no matter how useful it may prove within certain contexts, it appears doomed to failure. The research on teaching of the past decade has revealed the complexities of the teacher's task - that teaching is a "complex cognitive skill" (Leinhardt, 1989, p. 53) which has tended to defy experimental and correlational efforts to define and categorise it (Bertrand and LeClerc, 1985). Scriven's distinction between the "quest for knowledge" and the "improvement of practice" models for research (Scriven, 1983, p. 8) appears appropriate in this context. The moves in teacher research over the past decade towards the study of "expert/novice distinctions" (Berliner, 1986, Magliaro and Borko, 1986, Carter, Sabers, Cushing, Pinnegar and Berliner, 1987, Leinhardt, Weidman and Hammond, 1987, Strahan, 1989, Leinhardt, 1989 and Livingston and Borko, 1990), "exemplary practice" (Tobin and Fraser, 1988) and naturalistic studies of elements of the teaching process (Weade and Evertson, 1988, Mehan, 1989, Hansen, 1989) all recognise the critical value of the "wisdom of practice" (Tobin and Fraser, 1988, Leinhardt, 1992) in understanding the task of teaching.

From studies of "knowledge growth in teaching", Shulman describes teacher knowledge as consisting of distinct domains, distinguishing between areas such as "content knowledge", "pedagogical content knowledge" and "curricular knowledge" (Shulman, 1986, pp. 9-10). He describes the first as the "missing paradigm" which, at that time, had been relatively unexplored by research, the previous emphasis having been upon "how teachers manage their classrooms, organise activities, allocate time and turns, structure assignments, ascribe praise and blame, formulate the levels for their questions, plan lessons, and judge general student understanding" (Shulman, 1986, p. 8). His recommendation for greater research emphasis upon subject-matter knowledge has borne fruit, particularly in mathematics education, where several studies have focused upon teachers' knowledge and understanding of central mathematical concepts such as functions and graphing, which are relevant in the present context (Stein, Baxter and Leinhardt, 1990, Even, 1990). Such studies indicate, among other things, that poorly organised or represented knowledge leads to a narrowing of instruction in terms of providing a poor foundation for future work, an overemphasis upon non-essential aspects of the concept, and failure to capitalise upon instructional opportunities for fostering connections between concepts and representations (Stein, Baxter and Leinhardt, 1990, p. 639).

In general, research intended to make explicit the nature of "effective" teaching has tended to fall into two main areas of focus. The first concerns itself with the study of teaching *behaviours*, observing and documenting classroom practices which are associated with successful instruction, usually as judged by colleagues, supervisors or students, and measured against criteria which usually include success upon achievement-based assessment. The "exemplary teaching studies" of Tobin and associates at Curtin University in Western Australia over the past five years have tended to fall largely into this category (Tobin, 1987, Tobin and Gallagher, 1987, Tobin and Fraser, 1988, Tobin, Kahle and Fraser, 1990). Such studies focused early upon such specific factors as "wait time" between question and answer (Tobin, 1987), the role of "target students" who tend to monopolise teacher attention and reduce the cognitive demands made upon other students (Tobin and Gallagher, 1987), in addition to attempts to categorise the more general factors which influence and shape classroom practice and curriculum implementation (Tobin and Fraser, 1988). Such features as classroom management, the assessment system, the use of textbooks, in addition to time demands placed upon teachers to "cover the work" at the expense of student understanding or success - all were found to act significantly to reduce the cognitive demands of the classroom activities, and the effectiveness of instruction.

Later studies focused more directly upon identifying the practices of teachers recognised by supervisors and colleagues as "effective".

> The exemplary teachers had well-managed classes and were able to concentrate on establishing a productive learning environment. Each teacher viewed teaching in terms of facilitating student learning . . Each teacher had a stated belief that students created their own knowledge as a result of active engagement in learning tasks . . . In all cases, the exemplary teachers had a thorough and comprehensive knowledge of the content they were to teach. Furthermore they had a range of teaching strategies that could be used without a great deal of conscious thought . . . teacher expectations for student performance were high, consistent and firm . . . These teachers thought and talked about teaching approaches and were receptive to ideas for change. (Tobin and Fraser, 1988, pp. 91-92)

Clearly many of the conclusions reached in this study represent a shift away from the observation of practice and point towards the other broad field of research into effective teaching - that of teacher thinking (Mitchell and Marland, 1989). Studies such as the naturalistic investigation conducted by Magliaro and Borko (1986) demonstrated that effective instruction could not be defined simply in terms of teaching behaviours and strategies, but needed to take into account the cognitive processes which accompanied instruction. This study attempted to define relationships between classroom variables (such as participation, task structures, and time engaged on reading tasks) and student achievement in reading, contrasting two student teachers with their supervisors. The results were largely inconclusive for these factors - differences in student outcome could not be explained by the variables in question, but related more to beliefs about teaching and conceptions of their role by the participants (Magliaro and Borko, 1986; 133-135).

The recognition of teaching as a "complex cognitive skill" led naturally to studies founded upon the expert/novice distinction as the basis for studying and defining the thinking which is associated with successful teaching practice. Drawing upon studies of the thinking of experts in other complex cognitive fields (such as chess playing, note taking and solving physics problems), expert/novice studies of teaching investigated such factors as "mental scaffolding" used by teachers during instruction (Peterson and Comeaux, 1987), the introduction and integration of classroom routines (Leinhardt, Weidman and Hammond, 1987), processing and using information about students (Carter, Sabers, Cushing, Pinnegar and Berliner, 1987), agendas, lesson structures and explanations in mathematics lessons (Leinhardt, 1989), views of instruction (Strahan, 1989) and the planning and implementation of review lessons in high school mathematics (Livingston and Borko, 1990).

For all their diversity of focus, these studies are surprisingly consistent in their findings. Experts and novices differ in their recall, representation and analysis of classroom situations (Leinhardt, 1989):

> Expertise is characterised by speed of action, forward-directed solutions, accuracy, enriched representations, and elaborations of knowledge rich in depth and organisational quality, [and by] lessons that are open, flexible, responsive, problem-based and intricate. (pp. 73-74)

The phenomenon of "chunking" was frequently identified in the processes of experts (Strahan, 1989, Peterson and Comeaux, 1987) by which they were able to represent complex situations in simpler, more automated forms, described by Livingston and Borko (1990, p. 373) as "rich, well-developed, interconnected and easily accessible cognitive schema". Such cognitive organisation allowed successful teachers to recall large amounts of relevant classroom information (Carter *et al*, 1987), and quickly and effectively analyse quite complex classroom situations (Peterson and Comeaux, 1987, p. 321).

In terms of the SOLO Taxonomy, such cognitive organisation is represented by the difference between *multistructural* and *relational* levels of thinking. Whereas novice teachers tend to view the classroom situation as consisting of a large number of discrete and interwoven factors, experienced teachers seem able to draw relations between these diverse elements, and so to view the same scene as simpler in structure, and yet revealing of deeper meaning. Carter *et al* (1987) noted that experts seemed little interested in remembering much specific information about students - rather they merged the available information into a "group" picture (Carter *et al*, 1987, p. 150). Strahan (1989), using semantic ordered trees, found that experienced teachers constructed more complex and intricate representations of classroom events than novices, and expressed more student-centred views of teaching (Strahan, 1989, p. 64). Experts, too, made extensive use of classroom routines as a means of reducing cognitive load, and automating procedures by which effective learning could be facilitated (Leinhardt, Weidman and Hammond, 1987).

Such studies, while rich in descriptive and explanatory power, still fail to provide an effective means for improving practice. Just as the earlier studies of successful teaching strategies and behaviours do not provide the means by which novices or unsuccessful teachers might become “more expert”, knowledge of the ways in which experts think seems unlikely to fare any better in this regard. The transition from multistructural to relational classroom thinking cannot be accomplished easily; the very nature of teaching as a “complex cognitive skill” precludes the possibility of a “quick fix”. Similarly, in the context of the present study, it seems unlikely that the introduction of an innovation such as computer algebra software will result in changes to the ways in which the teachers’ cognitively organise their classroom interactions. The relevant question in this context concerns the likely effects upon experienced teachers when placed in a “novice” situation.

The implications of the preceding research suggest that the likely effects of such a classroom change will be upon teachers *pedagogic content knowledge* rather than their *content knowledge*. In fact, it is the relationship between these two which remains unclear. One of the points of focus in the proposed study will concern the effects of severing the links between *what* the teacher knows, and the ways in which this content may best be structured and presented in order for effective learning to occur. The research literature invariably lists expertise in subject matter as a necessary prerequisite for successful teaching

(Tobin and Fraser, 1989, Stein, Baxter and Leinhardt, 1990), but the nature of the relationship between the various forms of teaching knowledge remains unspecified. Computer algebra tools may be seen as a means of making explicit some significant aspects of these linkages.

The two approaches to research on teaching which have been examined above both contain elements of a “scientific” or “analytic” paradigm, the one focusing upon teaching behaviours and strategies, the other upon cognitive schemas and thinking. A third option exists which attempts to view teaching in a more global or holistic way, seeing it as a practice which bears many of the characteristics of art, rather than science (Zahorik, 1987), and which describes the knowledge of the successful teacher in terms of “craft” or “working” knowledge which is intuitive, difficult to verbalise and complex (Leinhardt, 1990, Gersten, Woodward and Morvant, 1992). Such approaches are often framed within a constructivist perspective, which recognises the importance of prior experience, existing beliefs and multiple kinds of knowledge in the understanding of teacher practice, and how teachers learn and change these practices (Sigel, 1984; Leinhardt, 1992, Prawat, 1992). It is perhaps instructive to note that proponents of the “teacher behaviour” model such as Tobin, and “cognitive scientists” such as Leinhardt are now working with these more global approaches to the understanding of the complex act of teaching.

The work of Tobin and colleagues in recent years, in particular, has moved away from the extensive study of classroom practices and strategies and towards the investigation of a particular aspect of teacher thinking associated with images and metaphors (Tobin, 1990, Tobin, Kahle and Fraser, 1990, Ritchie and Russell, 1991). Teachers' use of metaphors has been identified as a means by which many particular beliefs and practices may be categorised and, in some instances, changed. Such an approach is offered as potentially a means by which significant change in teacher beliefs and practices may be achieved (Tobin, 1992):

> Identification of salient teaching roles, and the metaphors used to conceptualize them, offers the possibility of changing what teachers do in the classroom. The metaphor used to make sense of a role is a master switch for associated belief sets of teachers . . . Reconceptualizing a role in terms of a new metaphor appears to switch an entirely different set of beliefs into operation. (p. 6)

Teachers who are encouraged to identify and critically examine their existing metaphors for instruction (which may include such descriptors as "captain of the ship", "entertainer", "policeman", "teacher as resource", "social director" and "travel agent") may then construct new metaphors which are perceived as more consistent with a desirable change in teaching practice. Thus, in the study by Ritchie and Russell (1991) an experienced teacher identified the metaphor of "teacher as travel agent" in such a way as to be more consistent with a constructivist mode than her current practice, developed and adopted the metaphor, and subsequently facilitated a change in instructional practice in the desired direction.

Such a model, then, offers the possibility of encouraging and facilitating teacher change in a way which is intuitive and appealing to practitioners, in that it does not involve identifying and attending to an array of variables and particular practices, but occurs on a deeper, less rational level. It seems likely that such an approach moves the focus of teaching from the *concrete-symbolic* mode of the SOLO Taxonomy to the intuitive, global *ikonic* mode. The existing research has already implied the importance of ikonic thinking by teachers, noting, for example, that "experts 'see' an entire scenario or episode before they act" (Leinhardt, 1989, p. 73), and recognising that, in addition to being a "complex, cognitive skill" teaching may also be conceived in terms of "improvisational performance" (Livingston and Borko, 1990).

The analysis and development of metaphors, then, may provide effective means by which preservice teachers may be assisted to think about changes in practice which are likely to effectively incorporate computer technology. Existing images and metaphors may provide important clues as to factors which may act to both encourage and inhibit such changes.