# *Eight*
# *Using the Tools*

The deliberate and goal-directed use of tools has always been an essential feature of human activity and a means by which we may most readily be distinguished from animal species. As means of enhancing, not only physical, but cognitive activity, tool use becomes a defining feature of that which is most uniquely human. Vygotsky begins his book, *Thought and Language* (1962) with a quote (in Latin) from Francis Bacon, broadly translated by Bruner (1986) as:

> Neither hand nor mind alone, left to itself, would amount to much. And what are these additional [tools] that perfect them? (p. 72)

Vygotsky is referring here to **thought** and **language** and the cognitive aids used to support these. Most importantly, language may be perceived as a cognitive tool which aids thinking (Vygotsky, 1978):

> Children solve practical tasks with the help of their speech, as well as their eyes and hands. (p. 86)

As described by Bruner, “language is... a way of sorting out one’s thoughts about things” (Bruner, 1986, p. 72). To Vygotsky, language is an example of a **sign system**. Sign systems include spoken and written language, and the number and symbol systems of mathematics - all used as tools to aid thinking. In fact, for Vygotsky, it was this use of sign systems which distinguished humans from animals (Vygotsky, 1978):

> Comparative analysis shows that such activity is absent from even the highest species of animals; we believe that these sign operations are the product of specific conditions of **social** development. (p. 39)

The use of tools serves to extend and enhance human capability, making possible that which otherwise may have been too difficult. At the same time, effective tool use requires a certain level of skill, both with regard to the use of the tool itself, and the object to be acted upon. In the case of the mathematical software tools considered here, effective use requires not only proficiency with regard to the computer application, but also prerequisite mathematical knowledge and skill. Having access to a computer algebra package, for example, will not permit students to act mathematically and meaningfully far beyond their current capabilities. Nor will they be in a position to choose a tool if they are unfamiliar with that tool or with the mathematical process in question.

Vygotsky's **Zone of Proximal Development** is particularly relevant in this context. While external factors (such as computer tools) may serve to extend and enhance individual functioning, the limits to which this will occur may be expected to be well-defined by the current state of the individual and by the nature and limitations of the tool. Computer tools for algebra serve two fundamental purposes - they permit **representation** and **manipulation** of algebraic ideas. While their representational capabilities are readily recognised as powerful and unique, the latter function appears to cross over boundaries set by centuries of mathematical tradition.

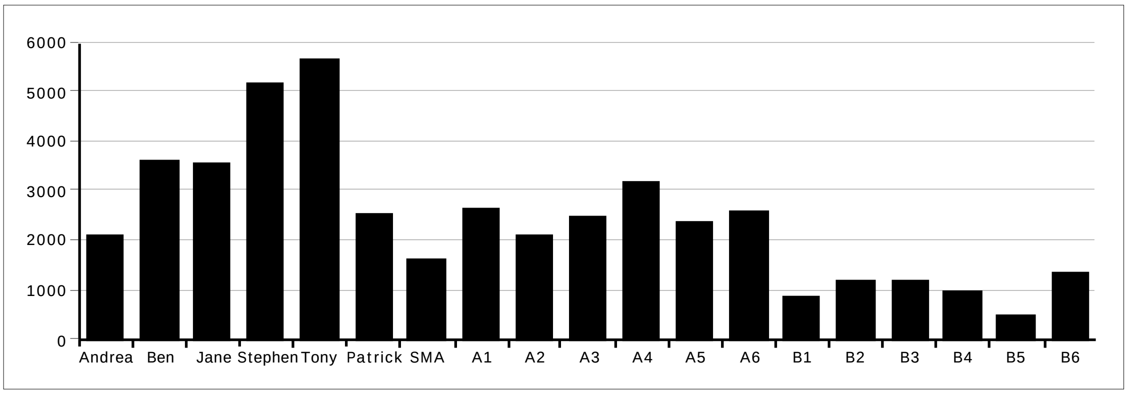
The tools used to enhance and enable mathematical activity have long been dominated by the sign and symbol systems by which this discipline is most readily recognised, aided externally by little more than pen, paper and, in certain circumstances, restricted access to geometrical tools. Mathematics as a purely cognitive activity has always been highly prized and nowhere more so than in schools. The advent of calculator and computer into mathematics learning has been met with a cautious and grudging acceptance, and the use of technology permitted only within very restricted boundaries.

This chapter examines the ways in which the participants engaged in the use of the various computer tools available to them within the confines of the study. It describes the context of this use, and the subsequent responses and reflections of the various participants. It seeks to make explicit those features of mathematical tool use which were most significant to the students and preservice teachers, and consequently to identify factors which may have served as both impediments and enhancements in their learning of algebra within a technology-rich environment. When considered within the context of the analyses of algebraic and pedagogical thinking described previously, this chapter sets the scene for the development of the grounded theory of mathematical software use which follows.

## Frequency of Tool Use

The individuals who participated in this study did so over varying periods of time, and engaged in a range of different activities. Consequently, a measure of the extent of their interaction with the available software tools must take this variance into account. A simple gauge as to the extent of the contribution of each individual to the data record may be derived from the number of **text units** for each, as determined by the qualitative analysis program, *NUD•IST*. Each text unit is essentially a single line of text, with up to eighty characters per line. *NUD•IST* automatically processes each document into such units, and numbers each for ease of analysis. A visual display of this measure is provided in Figure 8.1 (the numerical data for each of the graphs in this chapter is available in Appendix F). It is clear from this display that Stephen and Tony provided the most detailed and extensive information, and the Group B preservice teachers the shortest term of interaction. Much of the "bulk" of the research record may be attributed to the various research questions and tasks designed to provide the background information already examined. Actual tool use must be considered within this context.

FIGURE 8.1: Number of Text Units for Participants




The frequency of tool use in this study is measured in terms of what are labelled here as **incidents**. An incident of tool use involves the deliberate selection of a particular software application for a mathematical or pedagogical purpose. Incidents do not include occasions when a tool is selected but not acted upon, as occurred when some respondents were familiarising themselves with the program. The number of incidents ranged from 4 to 63, and are displayed in Figure 8.2. Relative to the number of text units for each respondent, however, the incidence of tool use across the participants is more clearly conveyed in Figure 8.3. While the students provide the most frequent examples of tools use in terms of raw numbers of incidents (as shown in Figure 8.2), the preservice teachers in Group B used the tools far more frequently when their more limited involvement is compensated for. Of the students, Ben, Andrea and Stephen were the most frequent users of available tools.

Figure 8.2: Frequency of Tool Use across Participants (Number of incidents)

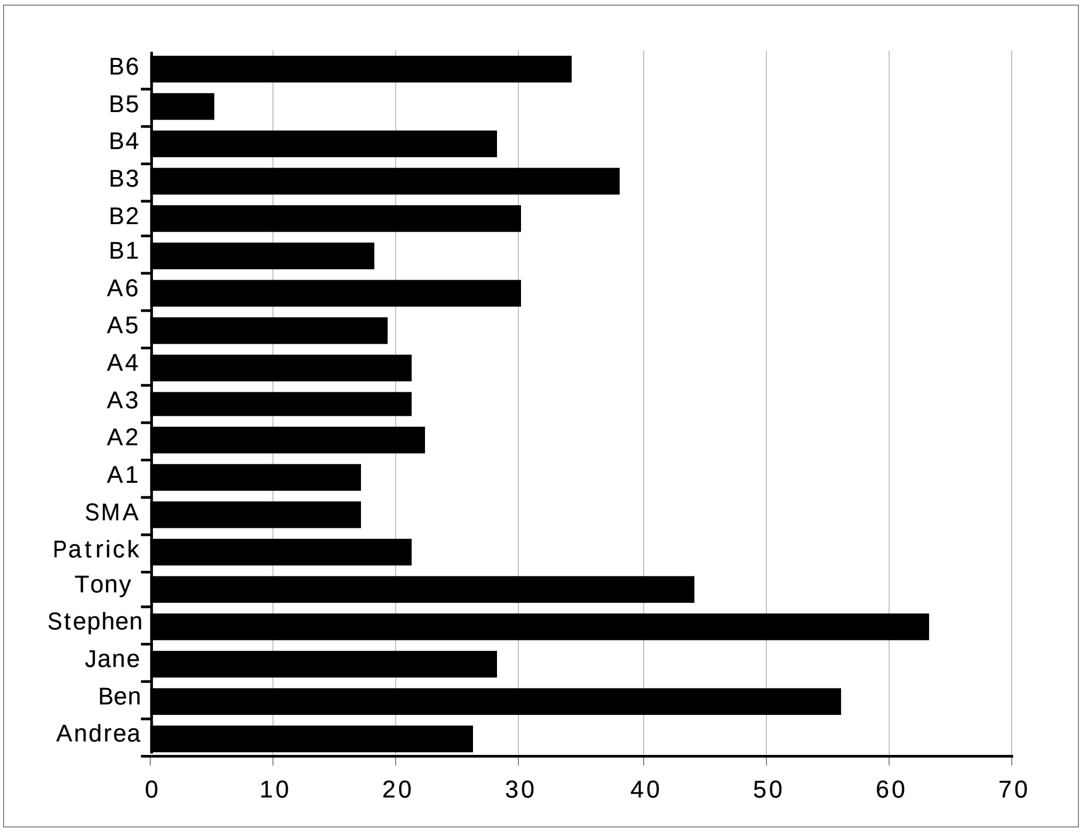
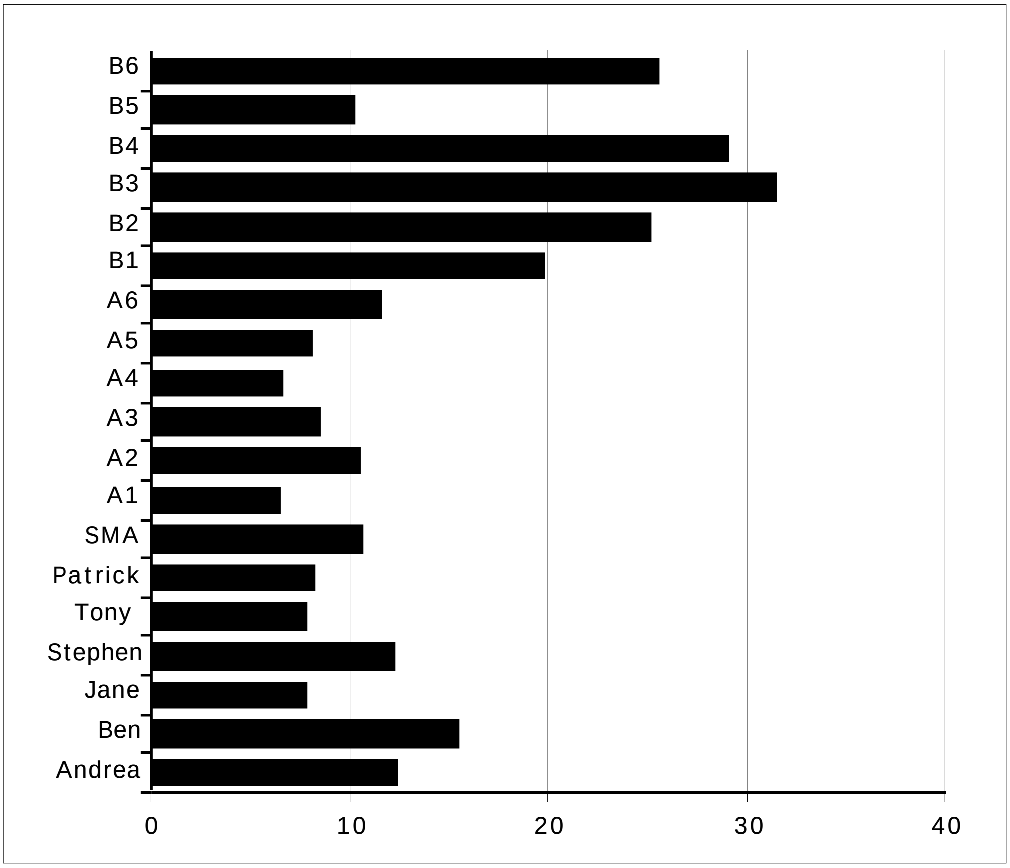
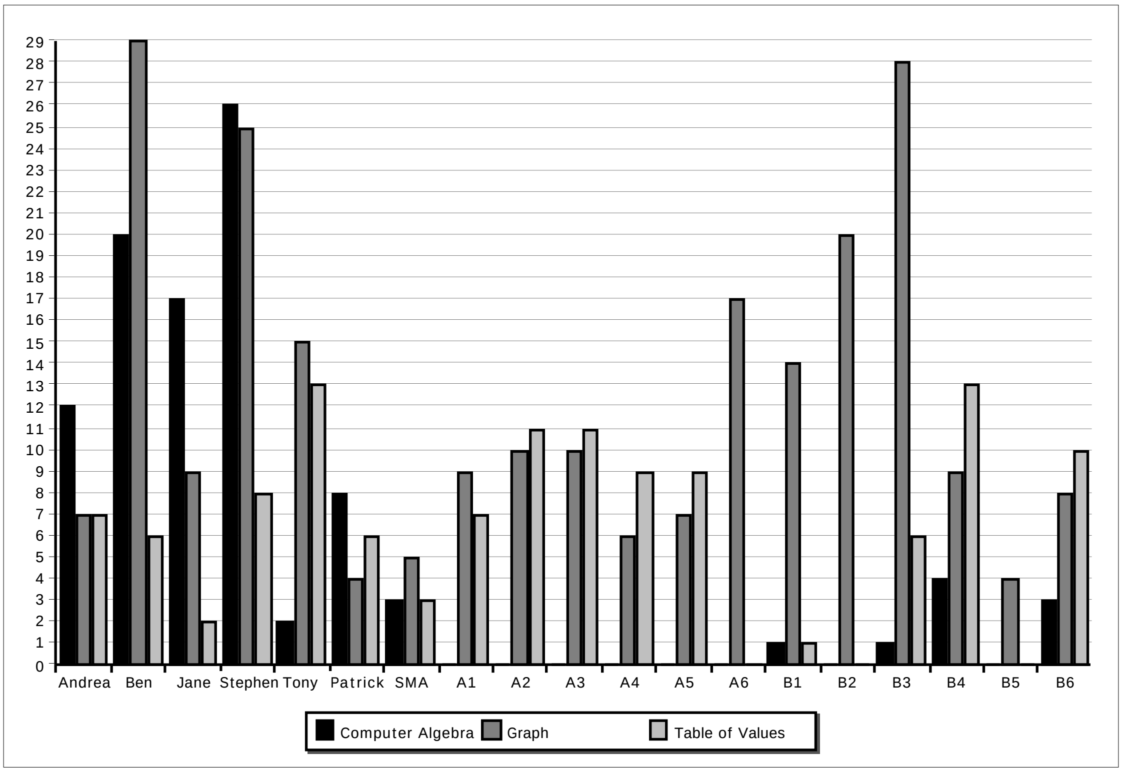


Figure 8.3: Relative Frequency of Tool Use across Participants (Percentage of total text units)




When this tool use is broken down into major software types (computer algebra, graph plotter and table of values), further detail emerges (Figure 8.4). The use of computer algebra tools among the Group B preservice teachers was minimal, even though this had been particularly emphasised in the introduction to the units of work (Group A did not have access to these tools). Among the students computer algebra use was far more common, but this occurred in the presence of the researcher who had made such use a particular priority for them. A similar emphasis upon the use of the table of values utility resulted in equally diverse results. For the students, the table of values was generally far less utilised than either graph plotter or computer algebra tool (although Andrea appeared to use graphs and tables equally) while preservice teachers appeared divided across the two groups - half showing some preference for the table of values over the graph plotter, and the rest displaying a strong preference for the graph plotter.

Figure 8.4: Frequency of Use of Tool Types




These varied patterns of usage across the participants will be examined in terms of two critical variables: the effects of mathematical **content** and **process**. The first considers the curricular context of the activity: whether participants were studying “Beginning Algebra” or “Calculus”, for example, and the nature of the tool use within this framework. The second considers the nature of the mathematical activities which accompanied and directed the use of available tools - which activities appeared to be most often associated with the use of computer tools? Additionally, the effects of the tool use will be examined through consideration of the respondents' own assessments - why they used the tools and how effective they found them to be.

**The Curricular Context**

Table 8.1 summarises the breakdown of tool use in relation to curricular content areas for the various participants. "A" represents the use of an "algebra" tool (*Theorist, MathMaster* or *CoCoA*), "G" represents use of a graph plotter, "T" stands for "table of values" and "√" indicates that the curricular topic was encountered but no software tools were used. It is clear from the table that the category labelled "Problems" was that most extensively encountered across all but the youngest participants. This grouping includes, not only those questions contained within the module, *Something to Think About,* but also open-ended and student-generated mathematical tasks which occurred frequently throughout the interactions. A clear focus for the group A preservice teachers was upon "early algebra" - the modules *Beginning Algebra* and *Equations and Problem Solving*, while the Group B preservice teachers were more engaged in problem-based activities. The breadth of curricular coverage by the various students may also be observed from the table.

Several features may be noted from this display. The use of the **table of values** was most common among the early algebra modules, where it represented an explicit priority. Similarly, the **graph plotter** was a central feature of the *Curve Sketching* module. Of the students, once again Andrea may be observed to give equal preference to the graphical and tabular representations; in contrast, Ben and Jane display a strong preference for the graphical representation, a preference shared by the

Group A preservice teachers, as evident in their approach to the mathematical problems they encountered. The Group B participants, however, appeared to make more effective use of the Table of Values in their problem-solving, but reverted to the graph plotter exclusively within the curricular contexts which they encountered.

Table 8.1

*Tool Use and Curricular Content*

| | Beg. Alg | Eqns | Coord Geom | Curve Sketch | Comp the Sq | Inv. Fns | Calc. | Review | Probs. |
|---|---|---|---|---|---|---|---|---|---|
| Andrea (S1) | GT | GT | √ | AGT | | | | AGT | AGT |
| Ben (S2) | | √ | G | AG | G | | AG | √ | AGT |
| Jane (S3) | √ | √ | AG | AG | G | | | A | A |
| Stephen(S4) | | | √ | AG | AGT | AGT | A | AT | AG |
| Tony (S5) | T | AGT | | G | | | | √ | |
| Patrick (S6) | AGT | A | | | | | | | |
| SMA (T1) | | | | | | | | | AGT |
| A1 | GT | T | | | | | | | G |
| A2 | GT | GT | | | | | | | G |
| A3 | | GT | | | | | | | G |
| A4 | T | GT | | | | | | √ | G |
| A5 | GT | GT | | | | | | | G |
| A6 | | | | G | | | | | |
| B1 | | | | | G | | G | | GT |
| B2 | | | | | | | G | | G |
| B3 | | | | | AGT | | AG | | GT |
| B4 | √ | | | | | | G | | AGT |
| B5 | | | | | G | | | | G |
| B6 | | GT | | | | | | | AGT |

The categories of **Problems** and **Reviews** provide the clearest insight into patterns of preferred tool use. Within several content-based modules, the majority of tool encounters were prompted explicitly. The focus within the *Equations* module, for example, was upon versatile thinking about equations, and users were prompted to access both graphs and tables specifically, and computer algebra if it was available. Within *Curve Sketching*, the graph plotter was the tool of choice. Use of other tools (as demonstrated by Andrea, for example) suggests initiative and cross-representational facility. Similarly, the use of tools within the *Review* modules was at the option of the user, and further demonstrates the extent to which the various participants had accepted the use of the computer tools as part of their mathematical learning experience. Ben, for example, worked through several review modules, and (although prompted at various points by the researcher) chose not to use any available tools. Andrea, on the other hand, displayed a willingness and, indeed, enthusiasm for the use of the tools which was unique among the participants. Although Stephen made effective use of a range of tools on various occasions, he displayed, like Ben, a reluctance to avail himself of their support, appearing to see it as fundamentally incompatible with his view of mathematics learning.

Ben notes, for example, that "I find most of maths OK - I can just bang it out, but when I don't know where something comes from or why it works I can't remember it". Ben was referring specifically in this case to rules such as the derivative of the natural logarithm of x being 1/x. He goes on to comment,

> Computers help me to visualise the question being asked. It also presents different methods to answering questions, e.g. if you didn't know the substitution or elimination method it can alter a question and make it easier to understand and easier to work out.

Ben attributes the active role in this encounter to the computer, which "presents" different methods and can "alter a question". In fact, he appears to be placing the computer tool in place of the teacher or tutor, who is expected to perform these functions, allowing him to adopt a passive role in his own learning. This passive role is further emphasised by Ben's reliance upon visualisation in his mathematics learning, in which he can simply "be shown" what to do and how to do it.

> Seeing the graph it verified my result - I am now 100% sure that 6 $^{2}/_{3}$ is the maximum and greatest speed is $^{4000}/_{27}$.
>
> [I used the table of values] to verify my results. Yes it was effective - it made me more confident of my answer... It was effective but I always find the graph more helpful - I like to visualise.
>
> The "functions of motion" model was helpful to visualise velocity over time - it's hard to picture when it's all just numbers.
>
> [I used the table of values] to compare two functions. It wasn't effective because I didn't pay any attention to it.

Ben used the computer primarily as a tool for **verification** of results which he had already obtained in most cases by traditional methods. While this is indeed a powerful role for algebra software tools, it is also an extremely limiting one. It places the learning emphasis firmly upon traditional methods, and the computer as an "optional extra". As a **purpose** of tool use, **verification** may be contrasted with **exploration**, with its implications of enquiry and student-initiated mathematical activity. Ben's comments suggest, too, a limitation of the tabular representation. While the graphical image is global and immediate, conveying information in an intuitive visual form, the table of values presents a relatively large amount of information in numerical form, which is more difficult to process and interpret. As Ben noted, if the user is not "paying attention", tabular information is easily overlooked. Ben's passive and visual approach to algebra learning would appear to work against effective use of tools such as the table of values, which require a more active analytical approach.

This distinction between **visual** and **analytical** use of computer tools appears significant in terms of understanding the preferences for different tools. While the graph plotter offers a fundamentally *visual* representation (in both the van Hiele sense and that of the SOLO *ikonic* mode of thinking), the table of values is essentially *analytical*, requiring quite complex information to be processed serially. This is not to imply that each tool cannot be used in the other mode. It was for this reason that the graph plotter developed for the study was enhanced to allow the coordinates of points to be displayed when the mouse is moved across the graph space. In particular, if a function has been graphed, the particular coordinates of the graph may be "traced" out using this feature, encouraging and supporting an analytical view of the functional information. Similarly, using the table of values to compare two functions allows an immediate visual response by the user as to whether the functions are identical or not. Stephen observes this feature when he notes:

> I used the table of values to compare two possible answers for a multiple answer question. It was effective as it showed the different results that each answer could obtain and how different the answers were.

In fact, for purposes of comparison, Stephen preferred the tabular representation to the graphical.

> I found the table of values most convincing; the graphs were helpful but not 100% sure ... To see what set of numbers were less than 5. Yes it was effective in doing this - probably more so than the graph since it just said true or false.

Stephen observed that the computer extended his mathematical competence but, like Ben, appears to value more his own ability to work unaided.

> We used the algebra tool to graph, differentiate and to find the minimum value of the question. This was effective as it showed the steps to use and in what order to find the answer. There were a couple of things I could not have done like factorising the large equations and finding the final answer in differentiating. To see the graph helped me to understand the answer better.

Jane, too, observed that the computer algebra tool extended her mathematical abilities.

> *[I used the computer algebra tool] to substitute into polynomials, solve simultaneous equations. [It] helped me to solve the problems that I couldn't do with pen and paper. I felt strange because I haven't used it before but I wouldn't mind using it again.*
>
> I used computer algebra for questions involving algebra and fractions and surds. We were simplifying and substituting and solving complicated equations. I found it to be helpful because it helped me to work out the questions. I think it will help me next time I get a question like that because I could see the steps involved and I feel I understand them better now.

Like Stephen, Jane points to another significant property of tool use in this context - the computer algebra tools made the mathematical process **explicit**. These were not programs which simply produced an answer; rather they involved the user in developing the solution, using their mathematical skills, and this feature was considered advantageous by the students. Ben and Stephen have already been noted as preferring to see **alternative approaches** to solutions, a further advantage perceived in the use of such tools (although subject to the intervention of the tutor).

Andrea reiterates several of these features of tool use when she notes that "we used [computer algebra] for recognising different types of equations and graphs, coordinate geometry and their uses. It was effective realising that there was more than one way to look at an equation".

At the same time, effective use of computer tools is limited by, among other things, the zone of proximal development of the user. The manipulative facility of computer tools is insufficient as a basis for understanding. As Patrick observed,

> When I used the computer to solve some equations I didn't learn anything because the computer did it for me, leaving me only with an answer, not the knowledge of how to do it.

Certain critical properties of tool use may be recognised at this stage. These include **depth** (visual or analytical), **purpose** (verification or exploration), **breadth** (the extent to which alternative approaches are available) and **process** (the extent to which the mathematical process is made explicit). Greater detail may be gained by consideration now of the relationship between tool use and mathematical activity.

## The Mathematical Context

The most common mathematical actions associated with tool use involved either **representation** or **manipulation**. The latter involves algebraic activities traditionally performed using pen and paper, but now available using computer tools, such as *Theorist* and *MathMaster*. Ten such actions were identified as occurring most frequently:

- Simplify
- Substitute
- Solve linear equations
- Solve non-linear equations
- Solve simultaneous equations
- Expand
- Factor
- Evaluate
- Differentiate
- Integrate

Table 8.2

*Tool Use and Mathematical Context*

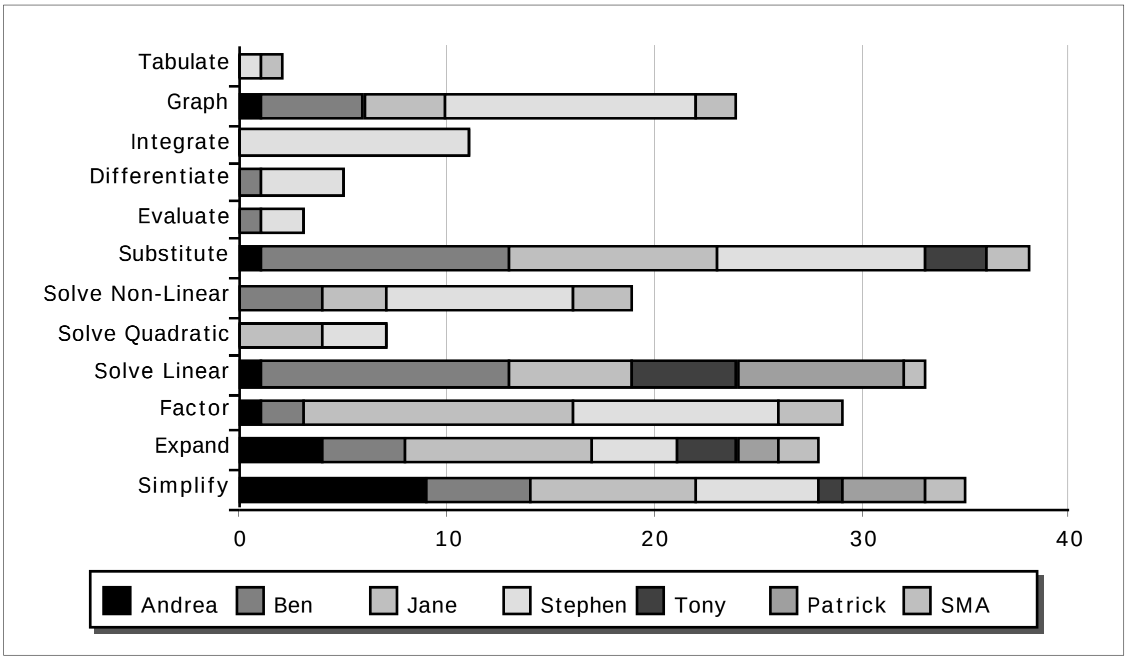
| | **Simplify** | **Subst.** | **Solve Linear** | **Solve Non-L** | **Solve Simul.** | **Expand** | **Factor** | **Evaluate** | **Calculus** |
|---|---|---|---|---|---|---|---|---|---|
| Andrea (S1) | A | A | AT | A | A | A | A | | |
| Ben (S2) | A | A | AT | A | AG | A | A | G | A |
| Jane (S3) | A | A | AGT | A | G | A | A | | |
| Stephen(S4) | A | A | T | A | | A | A | | A |
| Tony (S5) | A | A | AT | | T | A | A | | |
| Patrick (S6) | A | A | A | | | A | | G | |
| SMA (T1) | A | A | | AGT | | A | A | | |
| A1 | | | GT | GT | | | | GT | |
| A2 | | | GT | | | | | GT | |
| A3 | | | GT | | | | | T | |
| A4 | | | GT | | | | | T | |
| A5 | | | GT | | | | | T | |
| A6 | | | | | | | | | |
| B1 | | | | | | | | | G |
| B2 | | | | | | | G | G | G |
| B3 | | | | GT | | | G | AGT | G |
| B4 | | | | GT | | | A | A | |
| B5 | | | | | | | G | G | |
| B6 | | | T | G | | | | | |

Table 8.2 displays the tool use by the various participants which accompanied these mathematical actions. The use of multiple representations for equation solving was common for linear equations (since it occurred most frequently within the curricular context of the instructional modules, which required such use), but rare for non-linear functions (which formed the basis for several of the problems encountered). Similarly, the use of the graph plotter as a tool for factorising and calculus by several of the Group B preservice teachers was indicative of a high level of mathematical and technological sophistication - they were using the tool in inventive and strategic ways.

This notion of **strategic** tool use has already been identified as central to the concerns of this study. Such use is defined to be deliberate, goal-directed and insightful, and may be recognised as frequently versatile (involving the use of a variety of representations), active, analytical and displaying a repertoire of available and appropriate mathematical strategies. While my own use of the mathematical tools might be expected to reflect characteristics of strategic use, the evidence of Tables 8.1 and 8.2 suggests that Andrea, Stephen and several of the Group B preservice teachers may well be considered in this light. In terms of the curricular context, it appears likely that strategic use of the software tools may be limited to open-ended problem-solving situations, where the user must display initiative and some measure of inventiveness, rather than those content-based activities where the path to follow is largely predetermined.

Figure 8.5 provides a detailed breakdown of the pattern of mathematical use by the students of the computer algebra tool, *Theorist*. Because of its simple and intuitive interface and broad mathematical functionality, this was the preferred algebra tool for the project, and that most frequently used by the students. The most frequent mathematical actions for which the students used computer algebra were substitution, simplification, solving linear equations and graphing, although the pattern was different for each participant. These results are not intended to generalise beyond this sample, dependent as they are upon the particular focus and tasks posed within the study. In this context, however, they provide a valuable overview of frequency of use of the various mathematical capabilities of the program.

Figure 8.5: Frequency of use of mathematical functions of *Theorist* by students and researcher




For Andrea, the principal computer algebra activities were simplifying and expanding, with much less frequent use for factorising, substituting, solving and graphing. Since she also made quite extensive use of both graph plotter and table of values, this suggests that she was not dependent upon the algebra tool, but used it to complement the other available tools. Consider the following excerpts from Andrea’s text record:

> ************************
> Describe the difference between k*abs(x) and abs(kx) for constant k?
>
> For K*abs(x) the question is asking the absolute value of x the times by K
> but abs(kX) is asking for the absolute value of kx,. If K is a -ve
> number and was inside the function i.e. abs
> ************************
> Table : -4abs(x) : 5:05:24 PM
> 129 seconds
> ************************
> Comment: 5:07:55 PM
> For k*abs(x) and abs(kx): If k≥0 then k*abs(x)= abs(kx) if k<0 then
> k*abs(x) would be the exact reflection of abs(kx) i.e. abs(-4x)=4x but
> -4abs(x)=-4x => -abs(kx) = k*abs(x). Because k is negative, the RHS will
> be a negative graph.
> ************************

When confronted by a symbolic problem, Andrea turned to the table of values in order to consider it in numerical terms, which had more meaning for her than the symbols. By considering a single case (when k = 4) Andrea compared the two functions **4abs(x)** and **abs(4x)**, and then generalised from that. Although she used the table of values as her tool, she interpreted the tabular and symbolic results in graphical terms, as "reflection" and "negative graph". This is suggestive of strong cross-representational links. Her tutor suggested that she work through the section of the *Curve Sketching* module which dealt with the absolute value function in order to clarify some of her uncertainties in this regard.

At this point, Andrea encountered the definition of absolute value as **abs(x) =** $\sqrt{(x^2)}$. This was unfamiliar to her, and it was suggested that she investigate further. Her attention was also drawn to the possible distinction between $\sqrt{(x^2)}$ and $(\sqrt{x})^2$, in light of her previous consideration of **k*abs(x) and abs(k*x)**. To this end, she chose the *MathPalette*, entered the equation $\sqrt{(x^2)} = (\sqrt{x})^2$, and engaged in actions clearly suggestive of strategic software use.

```
* MathPalette© 12/9/94 4:36:32 PM
Table: 4:42:17 PM
55 seconds
Table:  sqrt(x ^ ( 2)) = ( sqrt(x)) ^ ( 2) : 4:42:19 PM
************************
Comment: 4:43:54 PM
The two functions are equal when x >= 0,and are not equal when x < 0.
When x >=0 then the values for x, y1 and y2 are all the same.
************************
function2  :  √(x^2)
function1  :  [√(x)]^2
************************
Comment: 4:46:25 PM
So when you square a number and then square root it, it becomes the
positive original number.
************************
```

Andrea offers an hypothesis, based upon her reflections upon the table of values result. She then turns to the graph plotter to further validate this conjecture.

HyperGraph : 4:48:00 PM
95 seconds
Function: (sqrt(x))^2
4:48:39 PM
************************
Comment: 4:49:27 PM
So the absolute value of x is the squared square root: abs(x) = √(x^2)
true for all values of x,while [√(x)]^2 = abs(x) when x ≥ 0.
************************
Comment: 4:51:48 PM
I expected √(x^2) to be a parabola, because from the table of values,
they were symmetrical.
************************

Note here that Andrea's visual image derived from the table of values had been incorrect. She had been misled by the presence of the exponent to expect a parabolic shape. Viewing the graph corrected this misconception. While this could have involved simply a surface viewing of the graph, Andrea analysed the graphical result in light of the symbolic form and was able to appreciate why the shape of the graph was linear rather than parabolic.

At this stage, Andrea was asked to consider an example of an equation involving the absolute value function. She turned to the graph plotter and observed the graphs of the two functions which made up the equation. Unsure as to whether she had found all possible solutions, Andrea then selected the computer algebra tool and used it to solve the equation symbolically.

MathPalette4:55:46 PM
7 seconds
HyperGraph : abs(2* x - 3) = x + 5 : 4:56:50 PM
abs(2* x - 3) = x + 5
************************
Comment: 4:58:43 PM
My confidence at this stage is about 90%. -2/3 I am 100% sure.
************************
Algebra Tool 4:59:26 PM
43 seconds

```
Theorist Student Edition : abs(2* x - 3) = x + 5 : 4:59:26 PM

(2*x-3)^2=(x+5)^2
4*x^2-12*x+9=x^2+10*x+25
3*x^2-22*x-16=0
3*(x-8)*(x+2/3)=0

abs(2*x-3)=x+5
abs(2*x-3)^2=(x+5)^2
abs(2*x-3)^2=x^2+10*x+25
```

Her ability to apply the method of "squaring both sides" using the algebra tool was illustrative of an action which was probably beyond her capabilities to successfully attempt unaided. *Theorist* offered her the support to explore with confidence an algebraic process which was new to her. Andrea used the available tools with deliberation and clear intent. They provided both manipulative support and enhanced representational facility which resulted in what appeared to be improved understanding of the concepts involved.

Stephen tended to use the available software tools to support and verify his own computations. Having access to a computer at home, he was given a copy of *Derive* which he was encouraged to use. This occurred rarely, and then only to view the graphs of functions, believing that the mastery of the manipulative aspects of algebra was a requirement of his course, and that there was little to be gained in having a computer perform these. Although he used computer algebra software often in his regular tutorial sessions (at the prompting of his tutor), such use was seldom spontaneous. He appeared to view using the computer for algebraic manipulation as a form of "cheating", although he was willing to use it to verify his own solutions, in the same way that he used the answers in the back of a textbook.

One aspect of tool use which became increasingly significant for all students as the collection of data proceeded was the perception of **confidence** in their solutions. Stephen, Ben and the others demonstrated that they were willing to accept answers in which they had, at times, less than full confidence, even while computer tools were available by which these results could be verified. This aspect of tool use became a major area of focus in the later stages of data collection, since it has important implications for students perceptions of mathematics learning, and their own responsibilities in this regard. It will be considered in greater detail later in this chapter, but Stephen provides an example of his reluctance to use the technology, even when his own skills may be insufficient to guarantee a complete or even correct result. The example arises from attempts by Stephen to answer a problem posed within the module, *"Something to Think About"*. The problem points out that the equations, $\mathbf{x = 2}$ and $\mathbf{x - 2 = 0}$ are considered to be mathematically equivalent statements, and yet squaring one produces an equation with two distinct solutions ( $\mathbf{x^2 = 4 \Rightarrow x = -2}$ **and 2**) while the other produces a single repeated root ( $\mathbf{(x - 2)^2 = 0 \Rightarrow x = 2}$). Stephen appeared unable to come to terms with the requirements of this question - he appeared to expect a question which was "well-defined" with a clear "right or wrong" solution. Eventually (after prompting from his tutor), Stephen used the graph plotter to study the graphs of the different functions and equations but was unable to offer an explanation with which he was satisfied.

The second part of the problem involved the equation $\mathbf{x - \sqrt{6 - x} = 0}$. He approached this almost with relief, as it appeared to fit more closely with his perception of an "algebra problem" - it signalled to Stephen that he should initiate equation solving techniques, beginning with squaring to eliminate the radical. This method proved unsuitable and Stephen's attempt at solution faltered. Unable to proceed, it was suggested that he might "use the computer" to help. His response was to choose the graph plotter and, later, the table of values:

Card: x = 2 and x-2 = 0? : 6:17:02 PM
Button: Hypergraph
Grapher: 6:25:56 PM
**************************
Comment: 6:28:14 PM
This shows that the function crosses the x axis at 2. About 80% confidence in 2 as a solution.
**************************
198 seconds
Table: 6:29:19 PM
**************************
Confidence now at nearly 100%
**************************

Since he still expressed some hesitation regarding his solution, it was suggested that he might use computer algebra to assist with the manipulation (and, in particular, he might try moving the radical expression to the right-hand side of the equation prior to squaring):

Button : Computer Algebra
Theorist:
• x - √(6 - x) = 0
Move √ to RHS:
x = √(6 - x)
Square both sides:
x^2 = -x + 6
Move all to LHS:
x^2 + x - 6 = 0
Factor:
(x + 3)(x - 2) = 0
Solve for x:
x = 2 and x = -3
Substitute x = -3 into quadratic: 9 = 9
Substitute into original: -6 = 0.

***************
Button : HyperGraph
Grapher: 6:33:05 PM
HyperGraph : x^2=6-x : 6:33:07 PM
135 seconds
************************
Comment: 6:35:36 PM
I found the table of values most convincing, the graphs were helpful but not 100% sure.
************************

Stephen verified his solution in two ways - first, by substituting both values derived from the quadratic back into the original equation (revealing that only one was a valid solution), and then again using the graph plotter. He nominates the table of values, however, as offering the most convincing evidence for the solution, while the algebra tool supported his manipulations. Although prompted towards the use of tools in this situation, Stephen's actions (and, in particular, the multiple verification of his results) suggest strategic use.

Further evidence of Stephen's reluctance to spontaneously use available tools (even when his own skills proved insufficient) is offered by a transcript of his attempt to complete the "Senior Algebra Review". It was emphasised prior to this review that Stephen should use the available tools if he was unsure of an answer - in fact, he would be penalised for incorrect responses. Even so, it took an incorrect result and his own confidence dropping to 50% before he chose to use the *MathPalette* tools to check a response. Stephen's hesitation regarding tool use is captured in the following comment:

> The down side of using computer tools is that in the test you can't use it, and you also learn the steps that you can do in the test. The steps that you have to do to get the answer to a particular type of question help you to get a feel for that type of question. When using the computer, you can see it doing it, but you don't think as much.
>
> On the plus side, using the computer has helped me by showing the easiest way to get to an answer and the setout of how to go about answering it.

```
Button :  Senior Algebra Review
Card : Quiz 1: Question 2  :   4:52:49 PM
************************
Comment: 4:53:35 PM
100%
************************
Button :  A*
Card : Quiz 1: Question 3  :   4:53:43 PM
CTRL-3 : 4:54:21 PM
************************
Comment: 4:54:21 PM
90%
************************
Button :  D*
Card : Quiz 1: Question 4  :   4:54:33 PM
CTRL-3 : 4:55:16 PM
************************
Comment: 4:55:16 PM
70%
************************
Button :  E
Button :  C*
Card : Quiz 1: Question 5  :   4:59:37 PM
CTRL-3 : 5:02:09 PM
************************
Comment: 5:02:09 PM
80%
************************
Button :  B*
Card : Quiz 1: Question 6  :   5:03:00 PM
CTRL-3 : 5:05:48 PM
************************
Comment: 5:05:48 PM
50%
************************
* MathPalette 30/8/94 5:06:15 PM
```

For Question 2, Stephen indicates confidence of 100% in his solution.

Although option “A” was the correct response, Stephen was only 90% confident.

Again, Stephen chose the correct response (D) but his confidence was now only 70%.

His first error (for which he had assumed 80% confidence).

His confidence now down to 50% for Question 6, Stephen selects the *MathPalette* as an aid to his substitution.

Although he possessed the mathematical competence and the knowledge and experience with the various tools to make use of them in a strategic way, it was Stephen’s perception of the existing mathematics learning culture which proved the major impediment in his use of the technology in a spontaneous and practical way. The use of the software tools by Stephen varied for the most part between what might be termed **passive** (initiated and directed by the tutor) and **reflexive**, in which tools were chosen but used in a superficial way.

The fact that Stephen failed to fully engage the problem itself is also a significant factor in this context. If mathematics learning involves a tension between **enquiry** at one extreme and **instruction** at the other, then Stephen appears to identify most readily with the latter extreme. "Mathematical" problems are those which are "well-defined" and possess a clear and attainable solution. Exploration is perceived as a "luxury" which time in preparation for the Higher School Certificate examination does not allow. Stephen's mathematical use of the computer tools available to him was dominantly representational (and strongly graphical in this regard); manipulative use was relatively rare, and restricted to those operations well beyond the scope of his abilities - large and difficult computations which were unlikely to be encountered within the limits of his examination preparation.

One further aspect of tool use which Stephen exposed in an early contact with a computer algebra tool (in this case, *Derive*) involved the capability of computer tools to make mathematical thinking *public*. The capability of the computer to make algebraic thinking explicit was an important element in the interview with Stephen which follows, in which the use of the computer algebra program was being demonstrated.

Friday, 7th May, 1993
(Stephen is given a few minutes to work on a solution to the question involving a cubic graph and its roots.)

I: (Gives the sheet with the graph of $f(x) = x^3 - 12x$ and four options to be judged true or false and sets the function up on the computer using DERIVE). Now the first question says that the maximum value of the function is 16. Your solution was . .

S: 65.

I: F(5) = 65 (from the graph) and that's obviously correct. (Enters F(5) on DERIVE) We put in F(5) and press S for simplify . . and out comes 65. Why did you use that? Um, why did you say that?

S: Um, because the maximum value is the positive in the x (indicates the given domain for the function, $-5 \le x \le 5$). And if it only goes . . if the range is between 5, you put in the 5 . . into the x values, and that will get you the maximum value on the y axis.

I: Right. So this is only in this sort of case, where it's a restricted domain?
S: Yes.
I: Is it always the maximum positive x value that gives you the maximum value of the function? Say you had a different graph. Would it be possible to have one where the maximum occurred at -5 instead of plus 5?
S: I don't think so.
I: So even if you had a graph that was the reverse of this one . .
S: Yeah . .
I: So the maximum occurs at the highest point? Okay, so we weren't tempted by that there (indicates the relative maximum on the curve) . It's obviously not the maximum is it?
S: No.
I: Okay, what if . . it wasn't a restricted domain? What if it didn't start at -5 and end at 5, it took all values? What would you say then about the maximum value of the function?
S: Um . . it's all reals on the y axis . .
I: But if we wanted the maximum value? Someone said, "What's the biggest . . the maximum value of the function?" What would you say?
S: (Pause) uh, you wouldn't be able to give one, because . .
I: It hasn't got one has it?
S: It's infinity.
I: That's right. Good, okay. . very good. Now next. It says, 'The equation f(x) = 0'. Now, from what we were just talking about, so that's an equation there . . f of x equals nought. 'The equation has exactly two roots, one negative and one positive.' And your answer?
S: No.
I: What do you mean by that - 'it's situated at zero'?
S: Um, because zero is zero on the axis, it's . . they can't have a negative or positive value.
I: Alright. So how many roots does this equation have?
S: Just positive.
I: So when you let f(x) = 0, it has . . what . . just one root at zero?
S: Yeah.
I: What about these two? What do you understand by a 'root"', when it's asking for the 'root' of an equation?
S: Oh, was that for this part (indicates the function in question)?
I: Oh, it refers to the same function, yes, sorry. f(x) is still x cubed minus 12 x. Let's see what it says here (types on the computer).. This time, we will just take f(x) = 0, and create an equation. And notice, because it's an equation, you don't have to use the dots in front: you just use an equals sign. The other one was defining a function, so the computer thinks there is a difference between functions and equations. It treats them differently. f(x) = 0. If we asked it to solve - you see the 'soLve' command, you just type 'L' to solve. Solve expression number 4. It says x = 0 is a solution. So is minus 2 root 3, and so is plus 2 root three. So that gives three . . three roots. So when we are looking at the roots of an equation, we are looking at . . what in terms of the graph?
S: The places it cuts through the x axis.
I: That's right. So in this case, the equation, f(x) = 0, has exactly . . how many roots?
S: Three.
I: Three roots. So that is false. You were right that it was false, but for the wrong reason.
S: Yeah, I didn't know . . I didn't know that was for this question.

The computer algebra program is being used as a tool for demonstration and verification. At the same time, it is visually exposing elements of Stephen's thinking which may have been incorrect. While he displayed some uncertainty regarding the nature of the question, his understanding of key terms such as "roots of an equation" is made public in this encounter, and the tutor is in a better position to judge areas of which the student may have been unsure. The interview continued, exposing further gaps in Stephen's understanding.

I: Yes, well that explains it. Alright. What about part C: it looks a bit mysterious. 'f(x) = k has three real roots for all real k.' What do you think now about this answer, now that you know that it's talking about this thing here (indicates the function) ?
S: Um . . yeah, I think it has got three roots the same as the other one.
I: For all real k?
S: Yeah.
I: What do you think the question is talking about when it's saying, 'f(x) = k for all real k'?
S: Um, if you put any number in for k, like, how many times will it touch the x axis?
I: Let's ask the computer (types in f(x) = 2). If we put in f(x) equals 2: Now remember, we just solved it for f(x) = 0, which is that picture there (indicates graph) where it cuts the x axis. Let me put in f(x) = 2, and solve . . takes a bit longer this one . . Now we don't have to get tied up in what they actually are, just how many of them there are ... and there's number three. Okay, there are three roots. They are pretty horrible looking things, but we could tell it . . just as an example, if I . . see the last command is 'approX': if you type 'x', it approximates, and so instead of giving it in that exact form, it will spit it out as a decimal. It turns out that one is actually equal to about three and a half. So each of those is just a decimal number. What would they be representing, those three numbers?
S: ... Where it crosses the x axis?
I: Where what crosses the x axis?
S: The curve.
I: Which curve?
S: This one (indicates the graph of f(x)).
I: But this is f of x. Here we are talking about f of x equals 2 in this case.
S: So, if you put . .
I: Go on.
S: If you . . make x cubed minus 12x equals 2 . .
I: Good, go on.
S: And then you factorise it . . you 'take the two over'.
I: Right, take the two across . . It's now equal to nought . . so it's now more like what we are used to, isn't it? So we could graph this thing. Let's do that. What you have done there is f(x) minus 2 . . right? And we know this is equal to x cubed minus 12x minus 2 . . now if we plot that . . we might actually zoom out a couple of times before we start... to make it bigger. See that little cross there: that's at (1, 1). So it's giving you an idea of the perspective of the graph. Right, so, now. if we press 'P' for 'Plot' it's going to plot f(x) - 2, or x cubed minus 12x minus 2 . . Now that's a function. We had probably better call it something else . . let's call it 'G'. As a function, it has a graph, which we will see in about ten minutes, when the thing gets around to doing it, right? And it's actually equal to another function, right? It's equal to f of x, take away two. Okay, what we are interested in is this

here, which is not a function, it is an equation. Right? By your definition, which is quite right. So what does this mean, in terms of the graph that will eventually appear up there?

S: It will cross through the y axis at negative two . .

I: Yes.

S: And . .

I: How will you solve the equation by looking at the graph?

S: Um . . put the x values in?

I: Okay, there is the graph . . now where are the solutions to the equation?

S: Um . . where it hits the x axis at (0, 0), and at the other two?

I: At the other two, yeah. It's not actually (0, 0), although we've zoomed so far that it looks like it. It's actually . . what it should be, of course, is . . this picture (f(x)) moved down two steps. Right?

S: Mm.

I: And as we saw, it's still got three solutions. So your theory is looking good so far, that we will always have three real solutions. Okay, we used k = 2 as an example. Let's try something different . . give me another number?

S: Minus 5.

I: Okay, so we are interested in f(x) = -5. To graph that, what would you graph?

S: x cubed minus 12x plus 5.

I: Okay, so it would be f(x) + 5. The same picture as . . that equation . . plot . . Now if using k = 2 moved the graph down two steps, what will you expect k = -5 will do?

S: Move it up.

I: Move it up 5 steps. Alright, while it's thinking about drawing the graph (it could take a while) . . so the values of k have what effect upon the graph of f(x)?

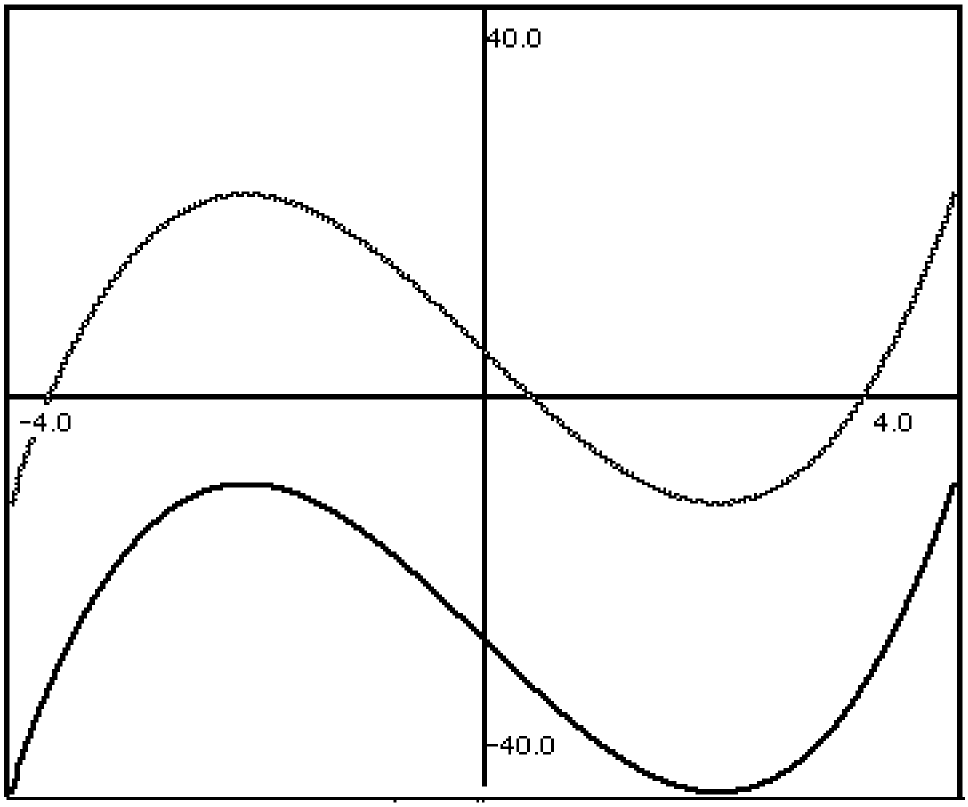
S: Move it up or down along the y axis.

I: Alright. Then we will ask the question again: do you think it's true that, for any value of k, it will always have three real roots?

S: Yeah.

This interview followed soon after Stephen had offered his definitions of function and equation, described previously. His confusion regarding the distinction between the two and his association with the visual format had prompted the interview focus upon these two terms. The interviewer is careful to point out examples of each, and to use the terms correctly in order to emphasise the similarities and differences between them. Stephen demonstrates good skills of graphical interpretation, recognising the effects of adding and subtracting a constant term upon the vertical position of a graph. He is, as yet, unable to visualise that certain vertical translations will result in the cubic graph cutting the x axis in less than three places (Figure 8.6).

Figure 8.6: Comparing $f$(x) and $f$(x) - 30




I: Alright. So it wouldn't matter if you put in . . k equals thirty? What would k = 30 do to the graph?
S: Move it up thirty . .
I: Oh, remember, positive numbers actually moved it down, and negatives moved it up. So k = 30 will move it down 30 steps.
S: Can I see . . (inspects the original graph closely).

Note that Stephen is analysing the graph at this point, alert to the fact that his previous assumptions are being questioned. He now realises that the graph may cross the axis at less than three points, and that, as a result, the equation may have less than three solutions.

I: f(x) minus 30 (enters on the computer) . . and because it takes so long to graph it, what we will do is solve it instead: the same effect. It's thinking about it . . like all good students, it doesn't rush into these things. So what's happening here is . . it's taking f of x (that picture there) and moving it down thirty steps. How many solutions is it going to have? How many places where it will cross the x axis? ...So what it means is that, as you saw, when you move it down far enough, this "hump" is going to be below the x axis and it will only cross in a single place, and only have one root. So it means that there are times when it doesn't have three roots. Will it always have at least one real root no matter how

far you move it up or down?
S: Yes.
I: Good, well at least it will if the graph continues forever. If we are looking at this particular graph, how far would you have to move it down so that there will be no real roots?
S: 65.
I: Yes, more than sixty five. Alright, we can see that it has three real roots for some values, one real root for other values, would there be anywhere where it could have two?
S: . . . (Long pause)
I: Can't imagine any? Imagine that we move this graph gradually down. Alright, as you move it one step down, when k equals minus one . . I'm sorry, k = 1, and 2, we move it two steps down; three . . all this time, it's still got three zeros, three roots. What about when k equals 16? . . What will happen at that point? And I'll point out that that's actually the height of that . .
S: It will just touch the x axis at this one . .
I: So it will have . . ?
S: Two.
I: So there is at least one place where there are two solutions. Anymore? Anywhere else that you could imagine it would have just two?
S: Move it up 16.
I: Yeah, that's right. Okay. We've got the picture, and eventually this will have it drawn . . What it's drawing there, you remember, is f(x) minus 2, when k equals two, so it moves it down two steps. Now, . . we will let it go. Okay, you're doing well. Last question then. The minimum value of the function is minus 65. What did you say?
S: Yes.
I: Yes? No problem then. It's true, isn't it? What they were looking for, of course, there, was for people to be distracted by these . . what are called "relative maxima" and "relative minima". It means that, in that little area there, it's certainly a minimum point, but over the domain, it's not an absolute minimum. Okay. You did well.

The computer served a vital role in exposing aspects of Stephen's thinking about functions, equations and their links with graphs. He found the experience valuable and continued to relate strongly to the graphical form. His use of the manipulative tools available, however, remained minimal.

While Stephen was almost stubborn in his refusal to seek computer-based assistance for his mathematical difficulties, the same could not be said of Ben. As a student, Ben preferred the role of "passenger" rather than "driver", rarely exhibiting initiative and accepting responsibility for his own learning. His preference for "visualisation" reflects his desire "to be shown" and corroborates his generally passive approach to mathematics learning. Like Stephen, Ben had extensive opportunities over a prolonged period to become familiar with the software tools. Like Stephen, Ben rarely chose to use the tools spontaneously, even when relatively uncertain of his own response. Unlike Stephen, Ben was not unwilling to use the tools, but rather lacking the initiative to actively select them.

Consider Ben's approach to the problem: "What is the value of c if the vertex of the parabola $y = x^2 - 8x + c$ is a point of the x-axis?"

> Firstly, I graphed the parabola y = x^2 - 8x minus the value of c. Then I saw the vertex of this solution as (4, -16). Then I saw for the vertex to be a point on the x axis the y value would have to equal 0, so the vertex would have to be (4, 0). So I had to move the y value up 16 values thus making the y intercept 16. Therefore the equation would equal y = x^2 - 8x + 16 showing the value of c to equal 16. The table of values was also used to see if these values were correct.

Unsure of how to begin, Ben was prompted by the tutor to consider the graph of $y = x^2 - 8x$. This was followed by the process described above, demonstrating that Ben had competent mastery of the available tools (he actually used graph plotter, table of values and, finally, *Theorist* to check the factored form of his answer). His description is replete with elements of visual imagery, used to advantage when he imagined moving the entire graphical object up by 16 units to obtain his answer.

Ben was happy to use computer tools whenever prompted, both to verify and to obtain solutions to problems. While Stephen used computer algebra only for those manipulations which were beyond him, Ben used it as a convenience, especially for tiresome and routine manipulations such as the solving of simultaneous equations, and even for solving linear equations (which he was quite capable of solving "by hand").

The use of the available tools by the two younger students was, not surprisingly, far more limited mathematically than that by the older participants. Tony used *Theorist* in particular as his preferred tool, enjoying the support it offered him for solving linear equations (which he had recently encountered at school). Like Andrea, Tony was willing to use such tools spontaneously to both verify his work and to support his computations. His attitude towards the computer algebra program is captured in a short transcript in which he is demonstrating its use to a peer (C):

T: Oh goody, it's Theorist. I'd better tell you about this [Looks through files]. This one looks good - it says 'Good intro'.
See, it can do this kind of thing.

C: It looks a bit hard.

T: Sorry, it just looks hard. You can type in all kinds of stuff like this - like 'k 2a outside 3 times 4 times 999, 456a then outside brackets to the square root of 999, 456a and now we'll try and solve it. We are really testing it here.

C: Simplify?

T: Simplify? OK. Let's simplify this one.

C: It probably can't be simplified.

T: You're right. We can't do that one either, so we'll close off this one. What did we just do? Oh, it's alright now. Let's do one of these - is that pretty or what? [referring to a three-dimensional graph].

Tony demonstrates, not only a very positive attitude towards the computer algebra program, but a willingness to use it to "play" with mathematics which would traditionally be beyond his capabilities. This element of **curiosity** was also evident in Tony's interest in the extension module on *Chaos Theory*, He chose to work through this module himself, and demonstrated what might be considered strategic use of tools in his exploration, moving from numerical values to graph to explore features of interest which resulted from different values of the variable "r".

```
* Tony ( ) Exploring Algebra session : 4:00 PM, Tue, 1 Nov 1994
Card : Exploring Chaos  :   4:00:40 PM
Card : The sounds of chaos  :   4:00:45 PM
Button :  CHAOS [repeated 9 times]
Button :  Plot
Button :  The sounds of chaos
Button :  Plot
Button :  CHAOS
Field : r-values  :  -1
Button :  CHAOS
Field : r-values  :  4
Button :  CHAOS [repeated 8 times]
Field : r-values  :  1000
Button :  CHAOS
Field : r-values  :  1000
Button :  CHAOS
Field : r-values  :  100
Button :  CHAOS
Field : r-values  :  10
Button :  CHAOS
Field : r-values  :  11
Button :  CHAOS
* Tony 1/11/94 4:03:12 PM
---------------------------------------------
```

Since Patrick had not studied any algebra at school at the time of the data collection, symbolic manipulation for him was a meaningless exercise. Although he was successfully taught to use *Theorist* to solve linear equations by moving terms, he “didn’t learn anything because the computer did it for me leaving me only with an answer not the knowledge of how to do it”. Patrick was far more positive after using the concrete algebra facilities offered within the *Mathpalette*, and after engaging in the *Think of a Number* game within the *Beginning Algebra* module. This activity engaged the user in the use of variables as generalisers, playing the traditional algebra game (*I think of a number, multiply by 2, subtract 5...*) using the table of values representation to operate, not upon a single number or variable, but upon a listing of several numerical values. In this way, the student sees that the process of acting upon a symbolic variable corresponds to actions upon a potentially infinite array of numerical values. After engaging in this numerically based exercise twice, the student then attempts the same process using symbols within a computer algebra tool. In Patrick’s case, the preferred tool was *MathMaster*, since the interface allowed the user to select an expression and then explicitly operate upon it using the four basic operations. To play the "number game", then, Patrick entered "n" and "2" and selected the multiplication symbol to produce "2n". He then entered "5" and chose "subtract" to produce "2n-5". Continuing the process eventually leads back to the original value, "n".

Patrick found this symbolic manipulation to be a meaningful experience after having grounded the procedure in numerical values, using the table of values representation. He demonstrated strategic use of the computer algebra tool, then, by developing his own *Think of a Number* game, using *MathMaster* to support his simplification. Once again, the algebra tool supported learning beyond that which would normally have been possible for this student, and contributed to both increased understanding of the nature and meaning of variables (as demonstrated in his definition), and also to improved skills of algebraic manipulation.

The students generally appeared to use the manipulative tools offered by the computer for purposes of verification and for convenience. The high incidence of acts of substitution reflects the ease which this mathematical operation could be carried out using the computer algebra tool, *Theorist.* Students had simply to "drag" the expression or value to be substituted onto the algebraic equation, and the result would be displayed. Similarly, the solving of equations was facilitated by the *Theorist* interface, allowing the user to "drag" terms across the "equals sign" in a physical simulation of that method of equation solving. The obvious preference displayed for this method of equation solving by the student participants reflects the fact that the software encouraged such an approach. Having access to such a facility appeared in no way to detract from their skills when working without the support of the computer tools.

A final feature of computer use which appeared significant for several of the students was that associated with entry of algebraic expressions and equations, and even of numerical expressions to be evaluated. Initial entry of algebraic forms into a computer application appears to be associated with a process of **reconstruction**, in which the user must **transfer** the visual stimulus of the algebraic expression or equation into a form which the software may act upon. This involves a deliberate consideration of the various components and their relationships which may act to move the cognitive level from a superficial visual mode to one which is more analytical. It was as a means of exposing such thinking that a palette was created by which algebraic forms could be entered, and which would make explicit the user's recognition of the various parts of which they are composed.

Students who were most competent in their algebraic skills appeared to have little problem entering algebraic forms using a variety of forms - the expanded text-format of several of the programs ("3*x^2-4*x+2), the simplified text-format developed for the *MathPalette* and instructional modules ($3x^2$ - 4x + 2, where the exponent was placed by simply using the option key with the desired number), or the point-and-click interface offered by the palette. Most participants found that the use of the palette was too slow and cumbersome, and preferred the simplified format, adapted from that offered by the programs *MathMaster* and *CoCoA*. All students demonstrated skill and familiarity with the use of their own calculators, which they were required to use in their mathematics classes. It was noted, however, that the entry of expressions in correct two-dimensional format offered by the *MathPalette* encouraged greater confidence in their solutions, since they could see all the components of complex numerical expressions. When evaluating the expression $\frac{2.5\sqrt{31.6}}{24.9}$, for example, Ben worked first with his calculator, and then entered and evaluated it using the *MathPalette*.

> Confidence in my first answer was about 80% because I am using my own calculator. Confidence now 100% because I saw the actual expression set out for me.

Ben observed that, when using a calculator, each term of the expression disappears after entry to be replaced by the subsequent term. The computer allowed the entire expression to be viewed, and so any errors of entry would have been visible.

Jane offers a response which may be generalised to each of the student participants regarding use of the software tools when she comments:

> They help you to get the answer. They show you how to work out a question. Out of computer algebra, graph plotter and table of values, I have found the graph plotter to be most helpful - it shows you what the graph might look like, If I had a program like the algebra program at home, I might use it, but more for the 2 unit type questions.

Jane's reference to "2 unit type questions" implies questions involving a high level of manipulative difficulty. It appears that the students in general found the most valuable aspect of using manipulative algebra tools to be in making explicit the solution process: "seeing the steps". The same could be said of a written solution, or of watching the tutor demonstrate the development of such a solution. However, the computer appears to offer one important advantage in this regard: it supports the students themselves in developing the steps, and so encourages their involvement as active participants in the process, rather than as spectators.

The mathematical use of available tools by the preservice teachers was quite different for the two groups. Nowhere was the distinction between **instruction** and **enquiry** made more explicit than in the involvement of these participants. As already noted, the Group A preservice teachers had chosen, almost entirely, to work through the modules associated with *Beginning Algebra*. The assessment requirements for this group explicitly rewarded their comments regarding the units of work and the tools. The result was that, in general, these participants engaged in the project as an evaluation exercise and their thinking and responses were predominantly pedagogical, rather than mathematical. The two occasions when Group A individuals encountered the problems in the module *Something to Think About*, their tool use consisted simply of viewing the graphs for the functions they encountered. Their use of the tools might be thought of as **random** as, clicking to move from page to page within the modules, they clicked a button here or there to view graphs which interested them. They then moved on without attempting to engage the question, in the same way that Stephen avoided confronting a question which did not appear amenable to a well-defined solution. The Group A preservice teachers worked through the units as required, dutifully using the tools required of them and observed positive and negative features of the program and accompanying tools, but at no point did they display mathematical interest or **curiosity** which would have provided a stimulus to explore the mathematical ideas which they encountered. They appeared to think like **mathematics teachers** rather than like **mathematicians**.

It was common for participants in Group A to describe problems which they encountered with both program and tools. The former were associated most often with perceptions of disjointedness and repetition in the activities for the different modules:

> At this stage I would like to comment on some of the [negative] features of the package. The setup of the package seems to be very disjointed. It does not flow from one item to another ... Some of the tools did not link in very well with the package... (A1)
>
> The graph facility was very time-consuming in the 'zooming out' process. The establishment of the point of intersection of the two graphs is quite hard to determine. (A3)
>
> The first module I worked through was Curve Sketching. I found it to be very useful, but it was also slow, repetitive and time consuming. (A6)

Although the use of the tools had been demonstrated for these students, it was common for them to encounter problems entering expressions using the palette, and using text-file entry methods. At the time that this data was gathered, the modules required algebraic expressions to be entered in "expanded" form - as "y = 4*x^2 - 3*x", for example, instead of the simplified form. In spite of explanation to this effect, several of the preservice teachers attempted to enter equations such as "y = 4x".

The problems encountered by this group in particular served to inform the continued development of the instructional modules and the accompanying tools, especially the improved interface allowing simpler and more intuitive entry of mathematical forms. Steps were taken to provide better links between the activities, and to provide easier navigation through the modules (these included the addition of a new menu from which users could move to any part of the program at any time, and so reducing the problems of "getting lost" within the modules).

The overall assessment of the program by this group was quite positive, allowing for the difficulties which they expressed. They remained enthusiastic regarding the potential of the computer as a tool for algebra learning.

> The use of the computer in the classroom is a way of encouraging and motivating students to learn algebra without doing all the textbook problems that are typically set in formal classes. (A2)

> Two areas that I thought the computer could help greatly, especially as a time saving device, was in the area of graphing functions and the area of tabling data. (A4)

> ...I see the main role of computer technology in supporting the teaching and learning of algebra as taking away much of the boring, unnecessary manipulative work that traditional approaches give to the students. In this way, they are promoting understanding and meaning rather than blind application of teacher imposed rules. (A6)

Other positive features recognised included the provision of meaningful context for algebraic ideas, the freeing of students to work at their own pace and the encouragement of discussion and group approaches. However, the Group A preservice teachers appear to concur with the students who perceive the principal advantages offered by computer tools as those associated with **motivation**, **convenience** and **representation**, rather than supporting and extending the traditional mathematical processes associated with manipulation of algebraic forms.

The evaluative comments of the Group A participants, then, proved highly valuable in terms of the ongoing development of the package and tools, and their perceptions were informative of the role of computer tools in algebra learning. It was clear, however, that this group had not engaged in the use of the computer as a *mathematical* tool. It was for this purpose that the Group B preservice teachers were included in the study and expressly encouraged to make problem solving a priority. The resulting incidents of tool use were varied, inventive and frequently strategic in nature.

Consider, for example, the attempts by B3 to answer the question from the module *Something to Think About* , involving the equation:

$$\left(x^2 - 5x + 5\right)^{x^2 - 9x + 20} = 1$$

Her first attempt involved repeated use of the graph plotter:

```
Button :  (x^2-5x+5)^(x^2-9x+20)=1
y=(x^2-5*x+5)^(x^2-9*x+20)-1
Grapher: 11:56:50 AM
HyperGraph : y=(x^2-5*x+5)^(x^2-9*x+20)-1 : 11:56:50 AM
252 seconds
y=(x^2-5*x+5)^(x^2-9*x+20)-1
27 seconds
Button :  (x^2-5x+5)^(x^2-9x+20)=1
y=(x^2-5*x+5)^(x^2-9*x+20)-1
Grapher: 11:57:23 AM
HyperGraph : y=(x^2-5*x+5)^(x^2-9*x+20)-1 : 11:57:23 AM
y=(x^2-5*x+5)^(x^2-9*x+20)-1
10 seconds
* MathPalette 16/9/94 11:57:38 AM
82 seconds
Grapher: 11:59:15 AM
HyperGraph : y=(x^2-5*x+5)^(x^2-9*x+20)-1 : 11:59:15 AM
y=(x^2-5*x+5)^(x^2-9*x+20)-1
41 seconds
* MathPalette 16/9/94 12:00:01 PM
HyperGraph : 12:00:03 PM
Grapher: 12:02:03 PM
HyperGraph : y=(x^2-5*x+5)^(x^2-9*x+20)-1 : 12:02:03 PM
y=(x^2-5*x+5)^(x^2-9*x+20)-1
11 seconds
* MathPalette 16/9/94 12:02:20 PM
```

This provides an example of what has been described previously as **reflexive** use of the software tools. Such use appears to have been encouraged by the facility within the instructional modules which allowed functions to be graphed by simply “clicking” on them. This appeared to induce at times almost automatic viewing of the graphs in a clearly **visual** way. Obviously frustrated, B3 moved on to other activities, but demonstrated persistence and determination when she returned to this question at a later time with an improved repertoire of tools, including the table of values and the inbuilt “solver”:

```
* MathPalette 23/9/94 12:14:34 PM
Solver: Searching for the roots of  (x^2-5x+5)^(x^2-9x+20)-1 from -10 to
10 ...
5 solutions of (x^2-5x+5)^(x^2-9x+20)-1 have been found at x = 1 ,2 ,3 ,4
,5
HyperGraph : 5 solutions of (x^2-5x+5)^(x^2-9x+20)-1 have been found at x
= 1 ,2 ,3 ,4 ,5  : 12:31:58 PM
27 seconds
* MathPalette 23/9/94 12:32:30 PM
Table: 12:32:36 PM
x = 1 ,2 ,3 ,4 ,5  : 12:32:36 PM
Table :   :   12:36:00 PM
Table : 5 solutions of (x^2-5x+5)^(x^2-9x+20)-1 have been found at x = 1
,2 ,3 ,4 ,5   :   12:39:53 PM
************************
Comment: 12:40:16 PM
this is the new revised comment to this silly question. we have found 5
solutions the original 1, 4 &5. but now we have found 2 more solutions.
These solutions occur  when we have (-1)^2 and this is when x=3 (by
factorising both the base and index
************************
Comment: 12:46:47 PM
the other solution is when we have -1 to the power of a positive  which
occurs when x=2.
************************
```

This extract offers what might be considered a definitive example of strategic software use. It is deliberate, goal-directed, persistent and insightful, making thoughtful use of the range of available and appropriate tools to not only derive a solution, but to verify this result using multiple sources. It was included as an example of a problem which was not amenable to graphical solution (Figure 8.7), and yet was immediately accessible using the table of values. Those participants who were restricted to the graphical representation were disadvantaged in such a case.

Figure 8.7: A difficult function to graph

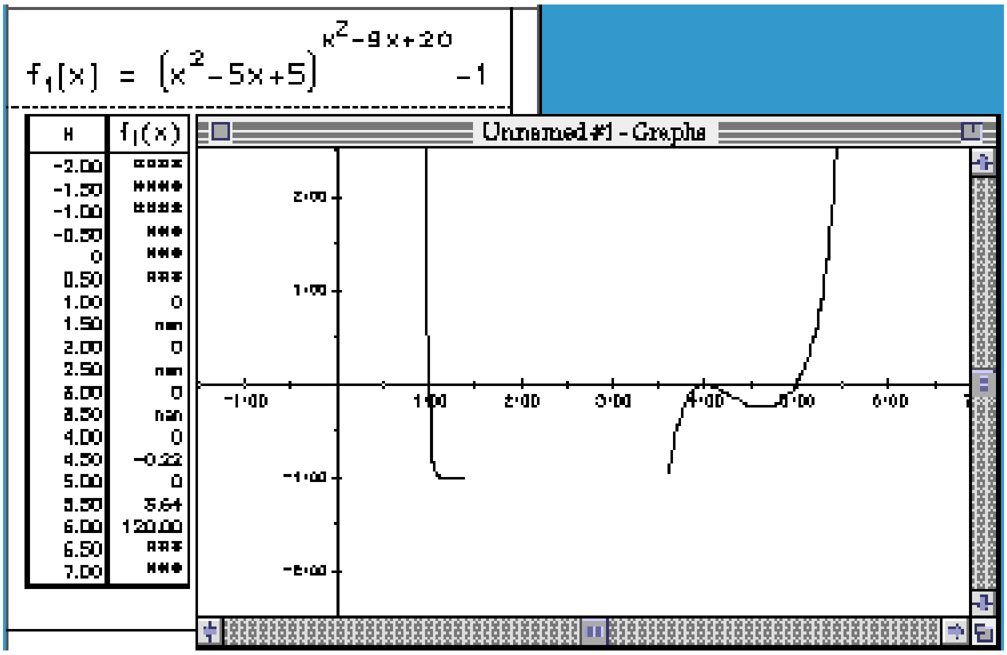
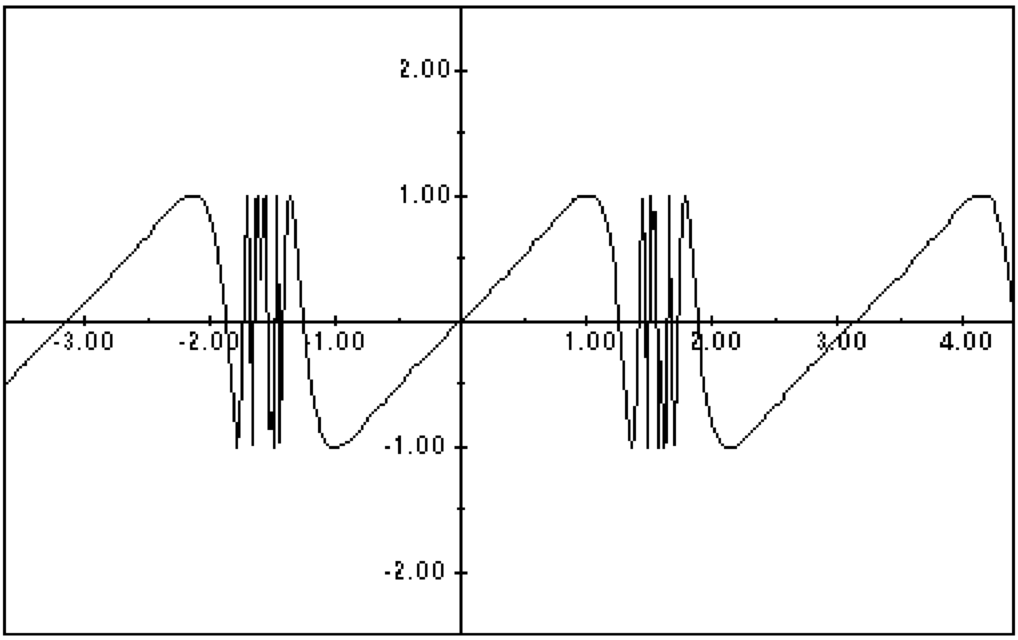



Figure 8.8: What is happening at the "fuzzy bits"?




When asked to describe "what is happening at the fuzzy bits" in the function **y = sin(tan(x))** (Figure 8.8), B4 used both *xFunctions* and the *HyperCard* graph plotter to examine the function, and then responded:

> CHAOS!!! is happening? As the sin and tan functions interact with each other. The function is periodic with a period which is large - wait and I will just find out how large...

She then returned to the graph plotter and used the "Trace" function to identify the period of the graph as 20 units. Like the function displayed in Figure 8.7, this problem required more than simply a visual use of the graph plotter to solve.

Even strategic use of the software tools, of course, does not guarantee a correct result. When B6 attempted a solution to the problem which Stephen had avoided, she used the graph plotter thoughtfully several times, and commented:

> Depending on how you square each equation you can get a different result. If you take x = 2 you can square it two ways: x^2 = 4 or x^2 - 4 = 0, for x - 2 = 0, the two ways are: x^2 - 4 = 0 or x^2 - 4x + 4 = 0 ... If we take the square of x = 2 to be x^2 - 4 = 0 and the square of x - 2 = 0 to be x^2 - 4x + 4 = 0 we find that the first equation has 2 solutions while the second equation has one solution.

Proceeding to the second part of the problem, she observes after graphing:

> If we square this equation we get x^2 - (6 - x)^2 = 0. This becomes x^2 - (36 - 12x - x^2) = x^2 - 36 + 12x - x^2 = 12x - 36 hence x = 3.

It seems likely that use of a computer algebra tool would have helped B6 avoid the frequent algebraic error, that $(a - b)^2 = a^2 - b^2$. It is also likely that her arrival at the correct solution using incorrect means was directly attributable to her use of the graph plotter, which allowed her to identify x = 3 as the required solution.

Whether complete or incomplete, correct or incorrect, the Group B preservice teachers engaged meaningfully and persistently with the open-ended mathematical explorations with which they were confronted. While the requirements of the assessment task undoubtedly contributed significantly to this interaction with both mathematics and tools, it does not entirely explain elements of perseverance and enquiry which appeared consistently. Rather it seems likely that **interest** and **curiosity** featured in the process.

Both groups of preservice teachers had chosen to be teachers of mathematics; they had recently completed extensive study of the subject and must be assumed to be at least competent in this regard and, presumably, interested in mathematics itself. Both groups interacted with essentially the same materials with basically the same available tools, and yet one group engaged meaningfully in open-ended mathematical exploration, while the other deliberately held back from such engagement. For Group A, it appears that "thinking like mathematics teachers" dominated their interactions with the tools and led to a failure to allow elements of curiosity and interest to surface. It is possible that, at least as perceived by one group of preservice teachers, mathematics is not something that mathematics teachers *do*, but simply something that they *teach*. This is consistent with a view of mathematics as a relatively fixed corpus of knowledge and skills which teachers, having acquired, are charged to pass on to their students, but not, it seems, to question or even to extend themselves.

My own use of the software tools, finally, involved exploring several of the same problems already described. I had attempted to select tasks appropriate and amenable to solution using computer tools by identifying activities which were unlikely to lie beyond the zone of proximal development for the intended users and yet which were not easily approached using traditional means. The problem displayed in Figure 8.7 was typical of such a task. It was not easily solved, either by traditional manipulative approaches or by simple viewing of a graph. At the same time, the solution was accessible, both using computer tools (in this case, table of values and the *MathPalette Solver* both produced immediate results) and by algebraic means, if the user approached the problem thoughtfully. In this case, the equation

$$\left(x^2 - 5x + 5\right)^{x^2 - 9x + 20} = 1$$

could be solved by observing that, for any expression of the form $\mathbf{a^b}$ to equal 1, either a = 1 or b = 0. This readily leads to the identification by quadratic methods of solutions of x = 1, 4, 4 and 5. This problem was drawn from the 1988 NCTM Yearbook, *The Ideas of Algebra*, in which these are listed as the solutions. I had accepted this result until I happened to use the table of values and found that x = 2 and x = 3 also appeared to be solutions. The *Solver* which I had developed to complement the *MathPalette* also produced solutions at values of 1, 2, 3, 4 and 5. As deduced by B3, the additional solutions arise when it is realised that, for $\mathbf{a^b}$, if b is even, then a = -1 will also produce a solution. This occurs only when x = 2 or 3.

This provides an example, then, of a question which is most appropriate to examine using software tools. The solution is not accessible only through extensive computational capabilities on the part of the computer, but also by traditional means. In fact, the computer directs the user back to such means to understand the result, and so leads to increased understanding and a useful learning experience.

I continued to investigate this function, using the range of tools available, in case any other features of interest emerged. Use of the "trace" function of the graph plotter, the "substitute" function and the table of values suggested that further investigation may be warranted around the value of x = 1.36.

Function: $(x^2-5x+5)^\wedge(x^2-9x+20)-1$
12:07:14 PM
Button : ZoomIn
Trace graph with optionkey - observe values around 1.36.
CTRL-2 : 12:11:10 PM
Substitute: ...
CTRL-2 : 12:11:22 PM
Substitute: $(x^2-5x+5)^\wedge(x^2-9x+20)-1$ ...
If $(x^2-5x+5)^\wedge(x^2-9x+20)-1$ ...x = 1.37 ... then the result is -1
CTRL-2 : 12:11:57 PM
Substitute: $(x^2-5x+5)^\wedge(x^2-9x+20)-1$ ...
If $(x^2-5x+5)^\wedge(x^2-9x+20)-1$ ...x = 1.36 ... then the result is -1
CTRL-2 : 12:12:27 PM
Substitute: $(x^2-5x+5)^\wedge(x^2-9x+20)-1$ ...
If $(x^2-5x+5)^\wedge(x^2-9x+20)-1$ ...x = 1.371 ... then the result is -1
CTRL-2 : 12:12:58 PM
Substitute: $(x^2-5x+5)^\wedge(x^2-9x+20)-1$ ...
If $(x^2-5x+5)^\wedge(x^2-9x+20)-1$ ...x = 1 ... then the result is 0
CTRL-2 : 12:13:24 PM
Substitute: $(x^2-5*x+5)^\wedge(x^2-9*x+20)-1$ ...
If $(x^2-5*x+5)^\wedge(x^2-9*x+20)-1$ ...x = 1.38 ... then the result is -1

Table: 12:14:53 PM
$y=(x^2-5x+5)^\wedge(x^2-9x+20)-1$ : 12:14:54 PM
Table : : 12:14:59 PM
Step Size : 0.01
function2 : $y=(x^2-5x+5)^\wedge(x^2-9x+20)-1$
Step Size : 0.001
function2 : $y=(x^2-5x+5)^\wedge(x^2-9x+20)-1$
Initial Value : 1.37
function2 : $(x^2-5x+5)^\wedge(x^2-9x+20)-1$
* SMA ( ) Exploring Algebra session : 12:19 PM, Fri, 28 Oct 1994
---------------------------------------------
* mathpalette 28/10/94 12:19:53 PM
Algebra Tool 12:20:08 PM
Theorist Student Edition : $y=(x^2-5x+5)^\wedge(x^2-9x+20)-1$ : 12:20:08 PM
************************
Comment: 12:24:18 PM
Well - another interesting one. $y=(x^2-5x+5)^\wedge(x^2-9x+20)-1$ has solutions at x = 1, 2, 3, 4, and 5, but the graph suggests a zero around 1.36 (found using the trace function) just as it becomes undefined. The table of values produces a value of -1 for all values from 1.36- 1.37.
************************

Questions such as this are extremely rich, mathematically, supporting and encouraging exploration by both teachers and students. Such questions demonstrate the potential for the use of computer tools to lower the barriers between teacher and students, offering the basis for them to become "co-learners" in mathematical exploration. While traditional questions associated with algebra learning tend to be closed, sequential and well-defined (and so discouraging the effective use of computer tools), questions such as those described here suggest that a critical aspect of "learning to use these new tools" will be in "learning to ask new questions". From the evidence of this study, strategic software use appears most likely to occur within the context of questions which are open-ended, rich in mathematical potential and yet accessible using both traditional and technological means. Within this study, such questions have included the high level problems mentioned above (most suitable for senior secondary and tertiary students), but may also be found among tasks available to younger students. For example, Ben engaged in a useful investigation - at the prompting of his tutor - as to whether $\mathbf{2x^2 - 4x + 2}$ should be classified as a "perfect square". In cooperation with his tutor, several different "definitions" of "perfect square" were recognised, which included:

- A numerical definition: a number which can be expressed in terms of two equal integral factors;
- A graphical definition: a quadratic function whose graph touches the x axis only once;
- A "factor" definition: a quadratic function which can be expressed using two equal factors;
- An "equation" definition: A quadratic equation with a single root (or two equal roots, depending upon your point of view).

As a result of this investigation, Ben came to realise that much of mathematics is subject to definition, and so whether a function may be considered a perfect square or not depends upon your point of view. He also gained considerably in his cross-representational facility, as he used graph plotter, number tools and computer algebra to explore the various representations associated with perfect squares.

At all levels, open-ended questions which are sufficiently challenging to require support and yet accessible enough to appear possible serve to encourage the type of software use which we describe here as strategic.

**Confidence and Uncertainty**

Finding the balance between tasks which are too difficult and those which are too easy has always been a hallmark of good teaching. Students invariably find the learning of algebra to be a challenging experience, its very symbolic nature placing demands upon formal thinking processes which are, for many high school students, still developing. It is not uncommon for students to spend a significant part of their time engaged in algebra learning in various stages of **uncertainty**. As has already been observed, students at all levels within this study were observed to be most comfortable with readily recognisable algebraic forms, especially equations which carried with them a signal to act in a specific predetermined way. Even such common forms as algebraic expressions (such as **4 - 3x**) proved a source of some uncertainty as students and preservice teachers were denied access to the action strategies available to them for equations.

When using computer software tools, these areas of uncertainty are likely to increase rather than decrease. In the majority of cases, the user is presented with a blank screen with little or no direction as to how to commence. In any given task, a student must first interpret the requirements of the task mathematically, deciding upon an appropriate course of mathematical action, but must additionally interpret the task technologically, choosing first between available tools and then, frequently, from a range of actions available within each tool. As the functionality of the tool increases, so does the range of potential choices and so, accordingly, does the level of uncertainty rise.

Particular software tools have attempted to address this problem. *Calculus T/L II*, for example, offers a unique support facility: having entered a function, equation, expression, data list or any of a number of possible objects, the program offers the user visible access to those actions appropriate to that object. Thus, for a function, for example, buttons for substitution, graphing, simplification, completing the square, differentiation, integration, limits and other mathematical actions are presented. Such an approach must serve to reduce the level of uncertainty experienced by the user, especially if that user is not mathematically sophisticated.

The interface offered by *MathMaster* proved advantageous for Patrick in developing his “Think of a Number” game. He first attempted to use *Theorist* to enter a series of mathematical operations which would eventually return him to the value of the original variable. The result was an expression of the form $\frac{2n - 6}{2} + 3$, which he found daunting. Using *MathMaster* Patrick was able to select each operation in turn, building up the expression by degrees, and having the program simplify it as he progressed, while keeping a visible record of each step in the process.

Within the instructional modules of the *Exploring Algebra* program and within the *MathPalette*, this problem was addressed directly through the inclusion of a **ToolKit** menu, which listed those mathematical functions available, from *Simplify, Expand* and *Solve* to *Animate, Areas under Curves* and *3D Graphs*. Selecting an option presents the user with a brief description of the available options and access to the tools which provide these. Selecting *Expand*, for example, the user is informed that expressions may be expanded using *MathMaster* or *CoCoA*, with an outline of how to access these features within these programs. Buttons are available for each which automatically open the selected tool, and the user may then carry out the desired operation. In this way, students are assisted in access to both the range of available mathematical actions and the available software tools which offer these.

The nature of the algebraic learning environment may be examined using constructs developed by Valsiner (1984) derived from Vygotsky's Zone of Proximal Development. Valsiner suggests that learning may be facilitated by first focusing the attention of the learner upon that to be learned by restricting the "Zone of Free Movement" (ZFM) and then encouraging desired actions which occur within a "Zone of Promoted Action" (ZPA). These elements feature strongly within the design of the computer-based instructional modules developed for this study. The computer environment serves as an inhibitory mechanism which actually promotes and encourages the use of the software tools as means of achieving desired mathematical actions. By making such tools familiar and easily accessible, students are encouraged to use them.

At the same time, the role of the tutor is a critical one. Wood's hierarchy of "levels of control" (Wood, 1986) suggests that the learner may be assisted across the zone of proximal development by judicious use of scaffolding by the tutor. Too much support and the learner fails to achieve independence; too little and the learner is overcome by uncertainty and becomes frustrated. Wood suggests that the support of the tutor be made *contingent* upon the response of the learner: each correct response by the student involves some withdrawal of tutor support, while each incorrect response initiates increased intervention. Wood defines five levels of control in this regard, from minimal intervention (involving broad general verbal suggestions) to actual demonstration by the tutor:

**Level 0:** No intervention

**Level 1:** General verbal prompt ("What else could you do?")

**Level 2:** Specific verbal prompt ("You might use your tools here.")

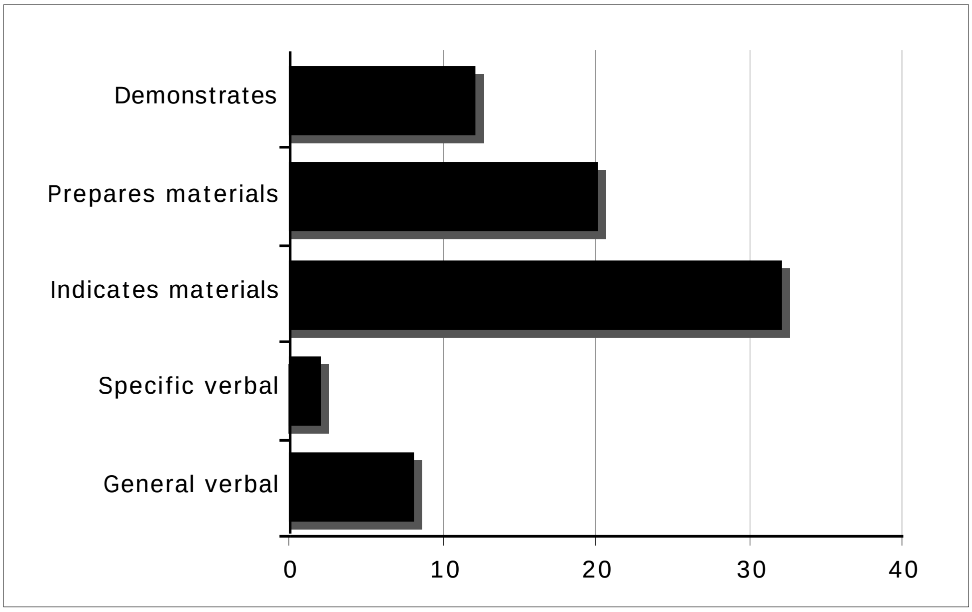
**Level 3:** Indicates materials to be used ("Why not use a graph plotter?")

**Level 4:** Prepares materials (selects and sets up tool for student.)

**Level 5:** Demonstrates use of tool.

This model was adopted by the tutor in his interactions with the students in this study. Although intervention was intended to be minimal, Figure 8.9 displays a clear tendency towards dominant control by the tutor. Note that examples of "non-intervention" are not included in this display - those incidents in which the students took the initiative in selecting and using the tools. Such occurrences were, however, relatively rare and the principal conclusion to be drawn from the results in Figure 8.9 centre upon the general reluctance by the students to make free use of the available tools.

Figure 8.9: Frequency and degree of tutor intervention




If **uncertainty** is perceived as occupying one extreme in the mathematical encounters of individuals, then **confidence** lies clearly at the other pole. Under what circumstances are students likely to be most confident in their dealings with mathematics, and what is the relationship of tool use with student confidence, particularly with regard to their own solutions? Increasingly the attention of the researcher was drawn to issues associated with student confidence after several participants commented upon their own perceptions of whether a particular response was or was not correct. Specific data were gathered which proved revealing of the relationship between confidence and tool use in the learning of algebra.

At various times each of the students engaged in “quizzes” or “reviews” as part of their progress through the instructional modules. These generally consisted of ten multiple-choice questions which simulated traditional assessment components which might be associated with particular topics. In order to further simulate “school-type” assessment, in most cases the software tools were not immediately available. Students were asked to attempt the questions themselves, unaided. They would then either select what they saw as the correct response or, in some cases, check their answer with selected tools before selecting from the given responses. As has already been mentioned, attempts were made to motivate the students to try to attain the highest possible scores: two marks, for example, would be awarded if their first response was correct, and a mark deducted for each incorrect response. It was sought to encourage the students to verify their answers (using the software tools if necessary) rather than simply offer their first response. Such validation appears compatible with a high level of responsibility for their own learning, and conducive to active participation in the learning process.

Consider the following excerpt from Andrea's record, in which she is attempting two questions from the "Elementary Algebra Review":

> Comment: 5:03:35 PM
> Expanding (x-2)(x^2+2x+4) I got x^3-8 first and then changed it to +8, then back to -8 (after seeing the answers). I am about 45% confident.
> ************************
> Comment: 5:07:02 PM
> After checking it on the computer I am now 85% confident.
> ************************
> Comment: 5:10:56 PM
> For (6x^4yz^2)/(-12xy^2z) I got (x^3z)/(-2y) and am 85 % confident.
> ************************

Andrea was permitted to use the computer tools before nominating an answer. Her confidence after checking these two questions rises, in the first case from 45% to 85%. Note, however, that even after checking, she is not 100% confident in her response. This pattern continued through the subsequent "Elementary Trigonometry Review", in which most (validated) responses again rated only 85% (and one 90%)

confidence, while those results which were not validated rated only 40-50% confidence. Clearly, Andrea is generally not confident of her answers to such questions. The fact that, even after validating answers using appropriate computer tools, she still does not express 100% confidence appears to reflect this general lack of confidence in her own abilities, rather than her perception of the computer tools as being inadequate.

The same pattern continued, even when attempting the “Beginning Algebra Review”, which involved questions which, while not difficult algebraically, appeared in a format which Andrea found unfamiliar. Later in this review, however, she appeared to become more comfortable:

```
I think it is y = 2x+3: 90%
************************
Button :  E*
Card: BA quiz 9
************************
Comment: 5:11:28 PM
y=3x-2: 3x-2=43 -> x = 15: Confidence 85%
************************
Table: 5:12:22 PM
function2  :  y=3x-2
************************
Comment: 5:13:12 PM
Confidence 100%: The computer proved to me that it is right.
************************
Button :  B*
Card: BA quiz 10
************************
Comment: 5:14:26 PM
100% because it is just like simplifying.
************************
Button :  A*
Card: Score Card
Button :  Back to the Start?
Andrea 25/7/94 5:15:20 PMThe score was 15
We used it to check my answer and to increase my confidence rating.  It
was effective, since after I plugged in the formula which I thought it
should have been and I saw that the values were correct.
```

Her comment that the "computer proved to me that it is right" suggests, not only faith in the technology, but confidence in her own ability to use the tool (in this case, the table of values) appropriately. As has been observed elsewhere, the table of values proves a most convincing representation.

Finally, Andrea's attempt to answer questions from the "Curves Review" with the assistance of the graph plotter saw her increase her confidence rating from 40% to 100% in one case, because "the computer proved it and I'm not going to argue with the computer".

The pattern of tool use was repeated consistently, not only for Andrea, but for each of the student participants: use of appropriate computer tools resulted in an increase in confidence in the proposed answer. For Ben , for example,

> For 8p^3+64 my CONFIDENCE is 100% because I did on the computer...
> My CONFIDENCE in solving the equation went from 80% to 100%...
> 1993: 2U Q1(a) 8.1 CONFIDENCE 50%, checked twice - confidence 100%

Stephen, too, while preferring to trust his own manipulative skill, nonetheless expressed an increase in confidence after using *Theorist*, on one occasion from 40% to 90-100%.

The evidence clearly supports the assertion that use of appropriate mathematical software resulted in increased confidence in the solutions presented by the students, even to quite complicated questions. A disturbing aspect of this data concerns the surprising frequency with which students expressed levels of confidence at or below 50% with regard to their answers to questions reviewing material with which they were familiar. Coupled with this lack of confidence in their own answers is the persistent failure of these students to seek to verify their results unless prompted to do so. Andrea proved the exception to this rule: she regularly and spontaneously availed herself of software support when she was uncertain of a result. For Stephen and Ben, however, their reluctance to make use of computer assistance, even when they expressed little confidence in their results, suggests again the influence of a culture of mathematics learning which devalues the use of external tools (the calculator appears to be the exception here) and within which the primary goal appears to be to produce an answer, whether correct or not. Students such as Ben in particular appear to view the responsibility for their own mathematics learning as residing with someone other than themselves.

## An Overview of Tool Use

It is now possible to view the incidents of tool use which occurred across all participants in terms of level descriptors which arise from the data. As previously described, specific examples of software use may be perceived as occupying positions at various points along a continuum, which may be described in the following terms:

**Level 0:** *Non-Use:* Although the software is available and appropriate, and the user has sufficient skill to use it, no use is made.

**Level 1:** *Passive*: The user is content for the tools to be operated by another, but takes no personal initiative.

**Level 2:** *Random*: Use is not goal-directed and bears no relation to the context.

**Level 3:** *Reflexive*: The user makes superficial and automatic use of appropriate tools.

**Level 4:** *Strategic*: Use of the tools is deliberate, goal-directed and insightful.

The relative frequency of occurrence of Levels 1 to 4 for the participants is displayed in Figure 8.10. The number of actual incidents associated with each level is considered as a percentage of the total number of incidents of tool use by each individual. Thus, of my own seventeen incidents of tools use, 4 were classed as reflexive (23.5%) and 13 as strategic (76.5%) .

The students were alone in displaying passive use, and showed no occurrence of random use - both owing to the presence of the tutor in every interaction with the tools. Incidents of strategic use by the students were more frequent than those of reflexive use (50% as opposed to 34%), while this situation was reversed for the preservice teachers, for whom the most common interactional type was reflexive use. As expected, Group B showed a greater tendency towards strategic use of the tools than did Group A, (23% of interactions as opposed to 13% for Group A).

Figure 8.10: Tool Use as Percentage of Incidents

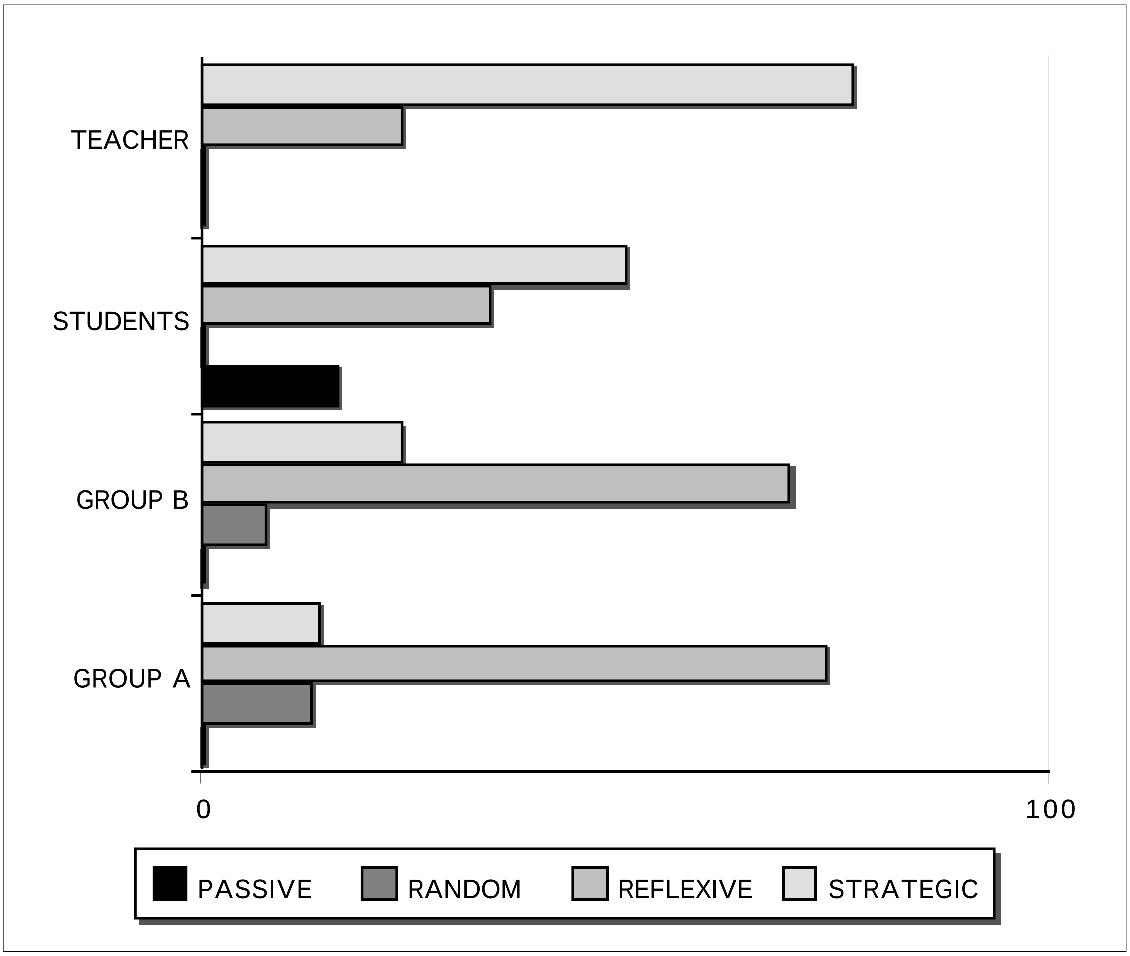


Figure 8.11: Patterns of Tool Use (Group A) - Number of incidents

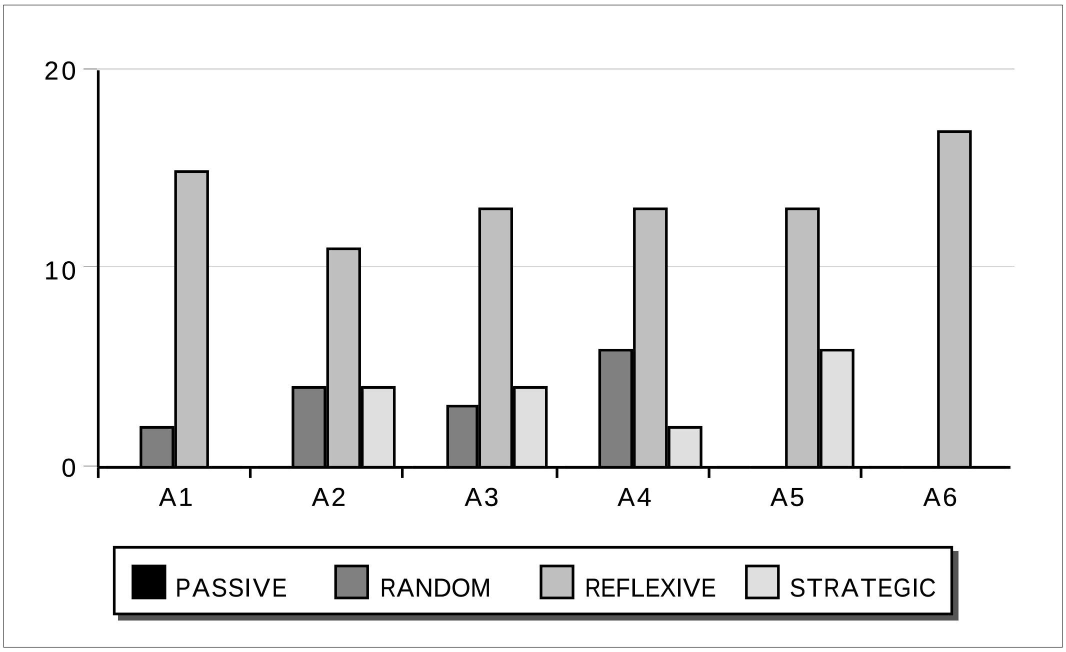
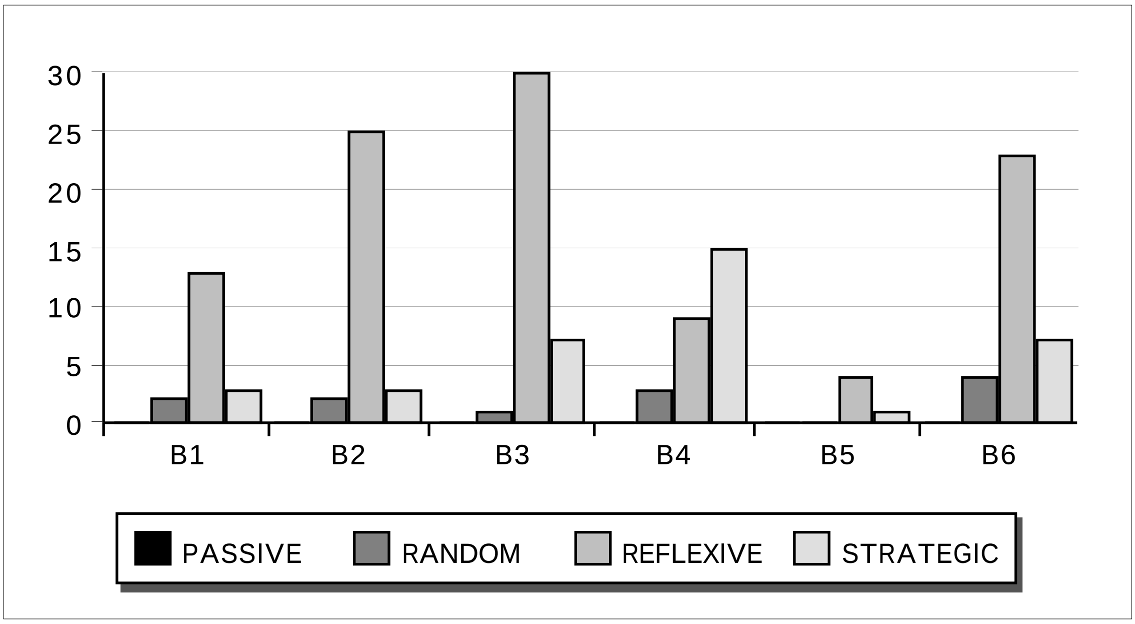
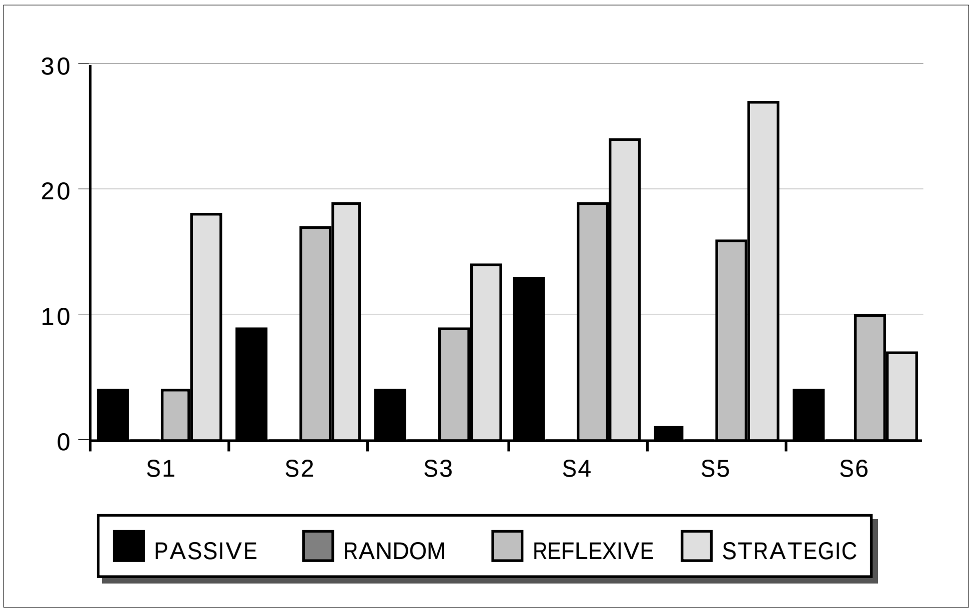



Figure 8.12: Patterns of Tool Use (Group B) - Number of incidents



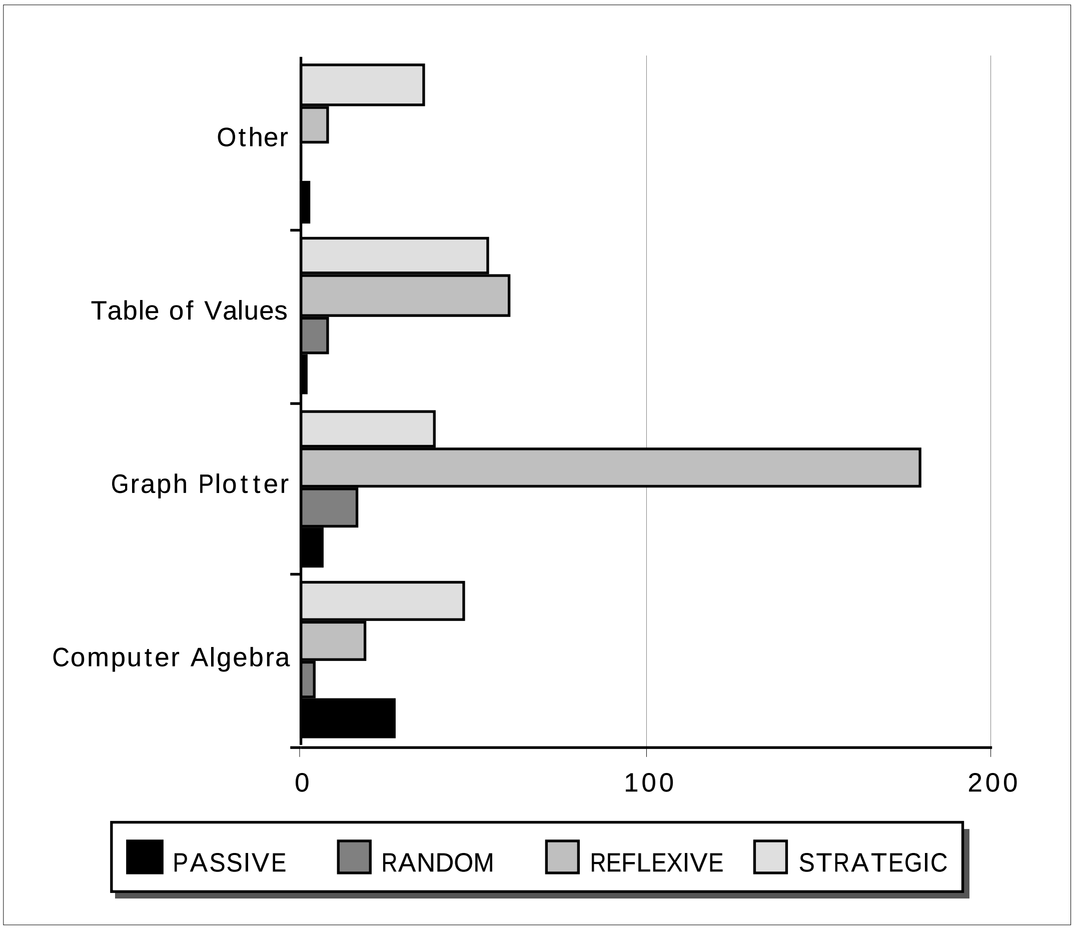

All Group B participants showed some strategic use of tools, while both A1 and A6 showed no evidence of such use. Tool use by both groups was dominated by reflexive activity, with some occurrence of random use in their early encounters.

Figure 8.13: Patterns of Tool Use (Students) - Number of incidents




The pattern of tool use for the students was quite different from that of the preservice teachers. While all engaged in some passive use of the tools (as the tutor demonstrated them), strategic use dominated the interactions for all but Patrick (S6). Andrea showed least reflexive tool use and Tony the highest incidence of strategic use (largely through his interactions with the utilities available within the *Exploring Chaos* module.) It is likely that the influence of the tutor must be considered as a significant factor in influencing the students in their high level use of the available tools; such was a specific and explicit priority of the instructional component of the study. The reluctance of most students to freely engage in tool use has already been noted; the fact that, in spite of this hesitancy, all students engaged meaningfully in mathematical interactions with the software must be considered a sign of success with regard to the technology-rich algebra learning environment created.

Figure 8.14: Breakdown of Tool Use by Tool Type




When the levels of tool use are considered in relation to the major tool types used for this study (Figure 8.14), several features appear significant. The high incidence of passive tool use for computer algebra reflects the priority accorded this tool type within the study. It suggests, too, that the other available tools were seen as requiring less specific instruction. The high frequency of reflexive use related to the graph plotter supports the observation that most use of this representation was of a visual rather than an analytical nature - users would quickly observe and draw required information from the graphical image and then move on to the next task. It is possible, too, that automating access to the graph plotter using *hypertext* facilities within the modules may have served to encourage reflexive tool use at the expense of more considered and analytical approaches.

Consideration of level of tool use in relation to content area adds further support to this criticism of the ease of access to the graphical representation. Those modules which most encouraged superficial viewing of graphs by simply "clicking" on algebraic forms as they occurred (*Curve Sketching, Completing the Square, Coordinate Geometry* and *Calculus*) were those most strongly associated with reflexive use of the tools. As has already been observed, strategic use appears to be most commonly associated with open-ended tasks which offer opportunities for **exploration**, while reflexive use most often coincides with those activities involving highly structured and predetermined **instructional** sequences.

Figure 8.15: Breakdown of Tool Use by Content Area