

12. GETTING STARTED WITH BEZIER CURVES: ALGEBRA AND GEOMETRY

Most of the curves and lines you see on a computer screen (including the letters you are reading now) are produced using Bezier curves, a mix of geometry and algebra that we explore in this activity.

These curves are defined by several points: moving the points changes the curve!

As well as bringing together those two important branches of mathematics, algebra and geometry, Bezier Curves can be great fun.

They allow students to be creative: a suitably challenging assignment may involve students designing their own figures like the one shown using several of these curves – and then working out their algebraic forms using parametric equations.

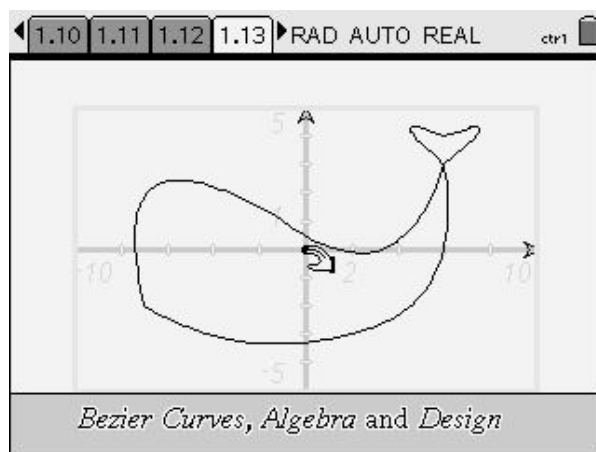


Figure 12.1: Fun with Bezier Curves

Step-by-Step Instructions

We begin with a new **Graphs & Geometry** page, and hide the Function Entry line using CTRL-G. Our first task is to construct a **slider** using a draggable point on a segment.

Choose **Segment** from the **Points & Lines** menu and place your segment somewhere out of the way, such as the top right corner of the page, as shown. Then use **Point On** to place a point anywhere along the segment. This will serve as our slider.

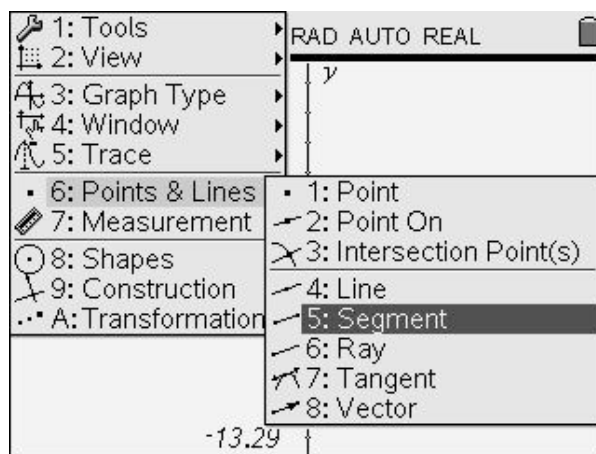


Figure 12.2: Creating a slider using the Segment tool

The slider needs to generate values between 0 and 1. Choose **Length** from the **Measurement** menu, click once on the draggable point on the segment, and then on the initial (left-hand end) point. Move this length somewhere out of the way and click again to drop it.

Return to the segment (still in **Length Measurement** mode) and click on the segment itself to measure the length. Move this result near the previous length and click to drop it. We now have the ingredients for our ratio.

Use the **Text Tool** to place the fraction “a/b” on the page as shown. Now choose **Calculate**, click once on the fraction, then follow the prompts: you will be asked to locate values for “a” and “b”: for “a”, click on the smaller length, and for the value of “b” click on the length of the full segment. Click to drop this result somewhere near the slider.

As you drag the control point on the segment, the values should vary between 0 and 1. Use the **Hide/Show tool** to hide the fraction and the two measured lengths, leaving just the slider and the ratio.

We are now ready to build a Bezier Curve!

Begin by using the **Point Tool** from the **Points & Lines** menu to drop three points around the graph page as shown. As you click to drop each point, press the SHIFT key and the letter you wish to name the point and each will be labeled as they are created.

To construct Bezier Curves we use the **Dilation Tool** from the **Transformations** menu. To create a dilation point we click on a start point (A), an end point (B) and then the ratio value produced by our slider. If the slider value is around one third (as shown), then a new point appears about one third of the distance between points A and B. Move the slider back and forth and our new point moves back and forth between A and B.

Repeat this process to put a dilation point between B and C.

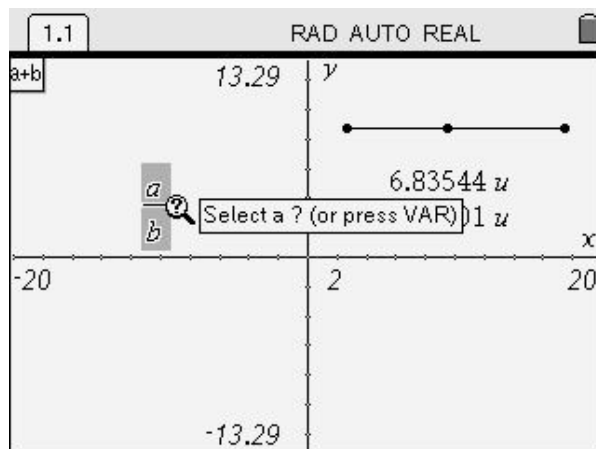


Figure 12.3: Calculating a ratio from our slider

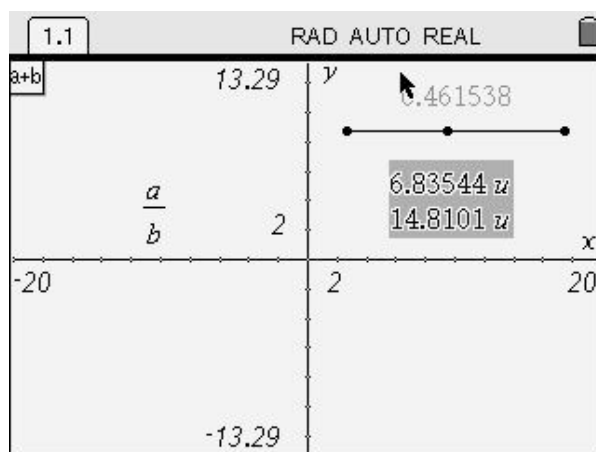


Figure 12.4: Calculating values for our ratio

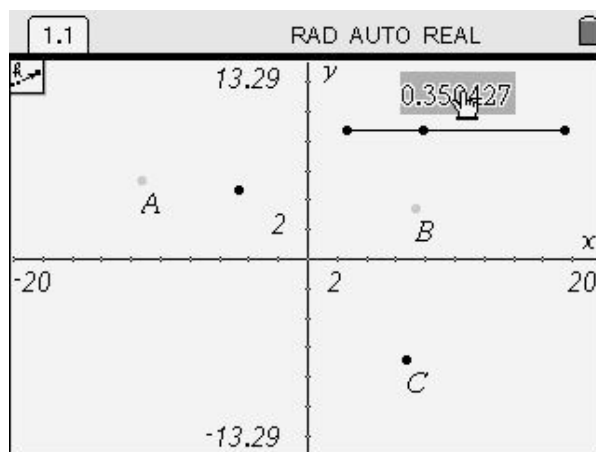


Figure 12.5: Creating a Dilation Point between A and B

Try the slider again: you should see both points follow the motion of the slider point. As the slider moves from 0 to 1, the first point should move from A to B, and the second point from B to C.

Finally, we create a dilation point between our two new points: after selecting the **Dilation Tool**, click first on the point between A and B, then on the point between B and finally, click on the slider ratio value.

Move the slider point back and forth, and you should observe the new point tracing a path that begins at A, curves around towards B, and ends at C.

Choose the **Locus Tool** from the **Construction menu** to see this path: click first on the final dilation point, and then on the actual slider point (not the value of the ratio this time) to see the path traced out by the dilation point as it is controlled by the slider point. This locus gives us our Bezier Curve – controlled by a start point (A), an end point (C) and a “shape point” (B).

While it is fun to build such curves geometrically, it is even more interesting to build them algebraically!

Insert a new **Calculator** page, and define a function called bez1 with three inputs:

Define $bez1(a, b, r) = a \cdot (1-r) + b \cdot r$

Can you describe what happens as the value of “r” varies between 0 and 1?

When $r = 0$, the function returns the value “a”; when $r = 1$, the function returns “b”. For values between 0 and 1, the function will return values between “a” and “b”. Try this with some numerical values: let $a = 2$, $b = 8$ and find the point one third of the way between these.

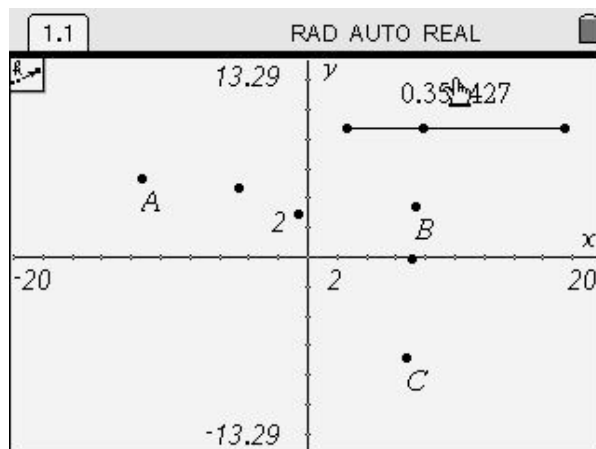


Figure 12.6: Three new dilation points

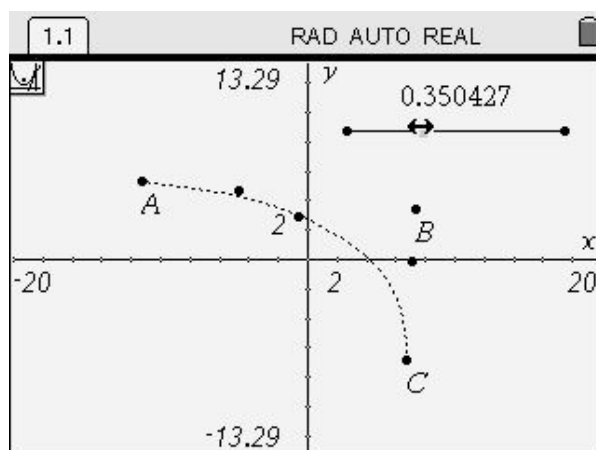


Figure 12.7: Locus of the path of our third dilation point

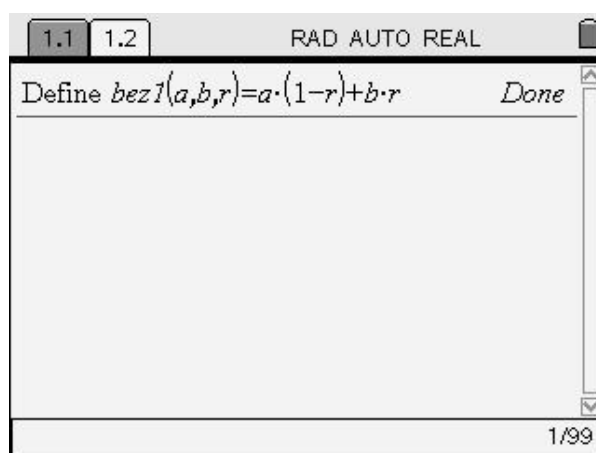


Figure 12.8: Defining a new function

This is the algebraic equivalent of what we just constructed geometrically!

Now we define a second order bezier function which takes four inputs: points a, b and c, and a ratio value r.

Define $\text{bez2}(a,b,c,r) = \text{bez1}(a,b,r) \cdot (1-r) + \text{bez1}(b,c,r) \cdot r$

Again, can you see the correspondence between this algebraic structure and the geometric construction? We have replaced our initial numerical values with the bez1 function, first giving the point between a and b, and then the point between b and c – and linking these in the ratio r.

Now to test our new function!

Return to the **Graphs & Geometry** page (CTRL-Left Arrow) and choose **Coordinates and Equations** from the **Tools Menu**. Click on point A to show the coordinates of that point, move away slightly and click again to drop those coordinates on the page. Repeat this for points B and C.

Click once to select the x-coordinate of point A, and press **VAR** to show the variables menu: press **ENTER** to **Store Var** and type “ax” to store the value.

Now repeat this process for each of the three coordinate pairs. The point A will consist of “ax” and “ay”, B will be (bx, by) and C (cx, cy). Each of the coordinate pairs will appear in **bold face** after they have been stored as variables.

Point to our locus curve and press **CTRL-MENU** to show the contextual menu: choose **Attributes**. Arrow down once and arrow right once to show the line as dotted. Press **ENTER**.

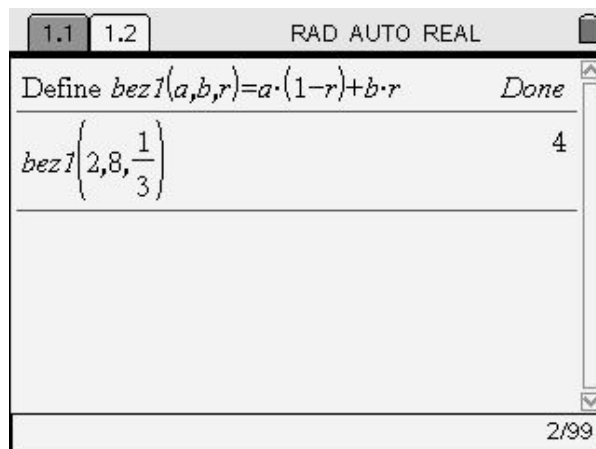


Figure 12.9: Testing our function numerically

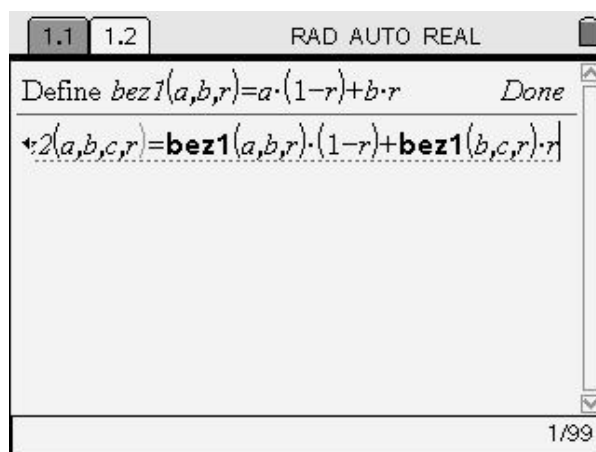


Figure 12.10: Defining a second order function

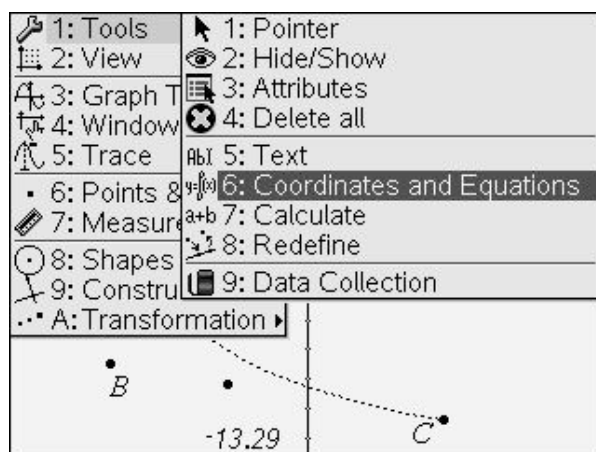


Figure 12.11: Storing our coordinates as variables

From the **Plot Type** menu, select **Parametric**, and enter our bez2 formula into the function entry line as follows:

$$X1(t) = \text{bez2}(ax, bx, cx, t)$$

$$Y1(t) = \text{bez2}(ay, by, cy, t)$$

Change the parameter values for “t” as shown:

$$0 < t < 1 \quad tstep=0.05.$$

Press **ENTER** and your dotted curve will be replaced by a continuous one, as the algebraic parametric form exactly matches the geometric bezier curve.

Since the coordinates have been stored as variables, you can even drag the points A, B and C around and the algebraic curve will change to match the geometric at all times.

For a further challenge, what about adding a fourth point D and building a “cubic bezier” curve?

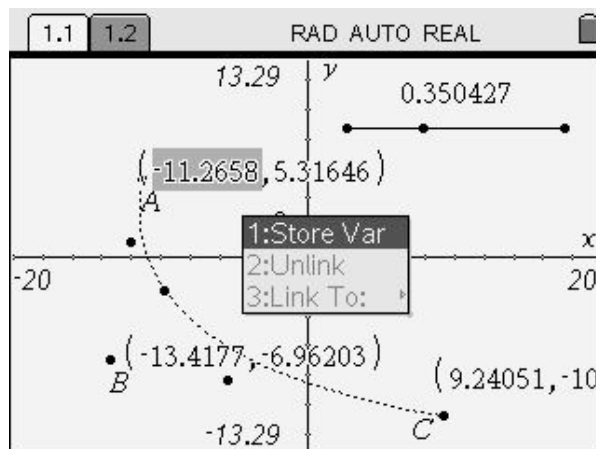


Figure 12.12: Storing coordinates as variables

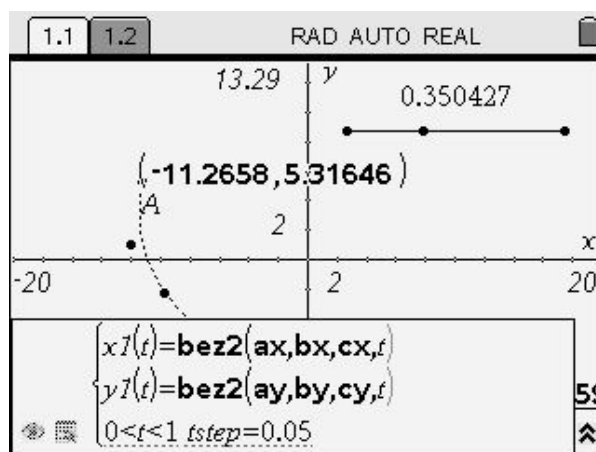


Figure 12.13: Parametric form for bez2

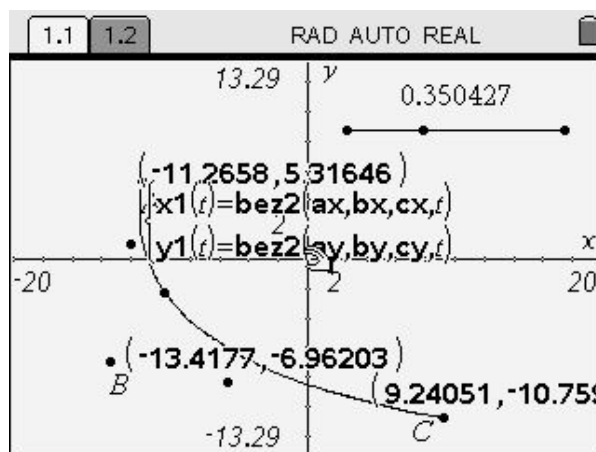


Figure 12.14: Algebraic Bezier Function

CAS Extension: If you are using TI-Nspire CAS, you can go to the **Calculator** and type the functions $x1(t)$ and $y1(t)$ to see the actual quadratic functions that define our curve.