Introducing the Integral Calculus: Integration By Parts

ID: XXXX

Time required
45 minutes

Activity Overview

In previous activities, students have explored the differential calculus through investigations of the methods of first principles, the product and quotient rules. In this activity the product rule becomes the basis for an integration method for more difficult integrals.

Concepts

- *Product rule of differentiation, areas under curves, integration methods, integration by parts.*

Teacher Preparation

This investigation offers opportunities for review and consolidation of earlier key concepts related to standard integrals and the product rule for differentiation, and for mastery of the method of integration by parts, supported by prepared programs and by the computer algebra system (CAS) available with the TI-Nspire CAS. As such, care should be taken to provide ample time for ALL students to engage actively with the requirements of the task, allowing some who may have missed aspects of earlier work the opportunity to build new and deeper understanding.

- *This activity can serve to consolidate earlier work on the product rule and on methods of integration. It offers a suitable introduction to the method of integration by parts.*
- *Begin by discussing approaches to more difficult integrals and review the methods used for differentiation. It is advisable that students have some experience with substitution methods for integration before attempting this activity.*
- *The screenshots on pages X–X (top) demonstrate expected student results. Refer to the screenshots on page X (bottom) for a preview of the student .tns file.*
- ***To download the .tns file, go to http://education.ti.com/exchange and enter "XXXX" in the search box.***

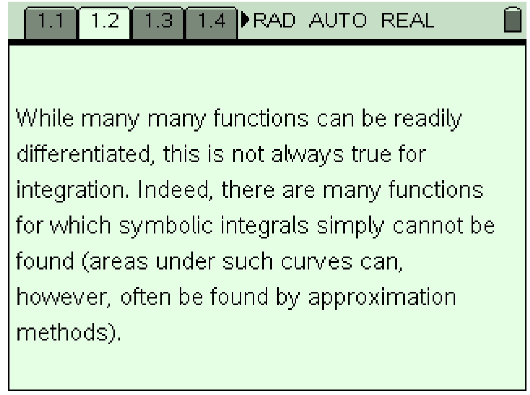
Classroom Management

- *This activity is intended to be* ***teacher led.*** *You should seat your students in pairs so they can work cooperatively on their handhelds. Use the following pages to present the material to the class and encourage discussion. Students will follow along using their handhelds, although the majority of the ideas and concepts are only presented in* ***this*** *document; be sure to cover all the material necessary for students' total comprehension.*
- *Students can either record their answers on the handheld or you may wish to have the class record their answers on a separate sheet of paper.*

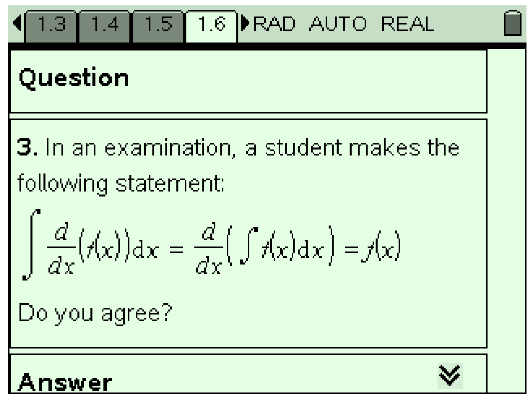
TI-Nspire™ Applications

Calculator, Notes, Lists & Spreadsheet, Graphs & Geometry and Programming.

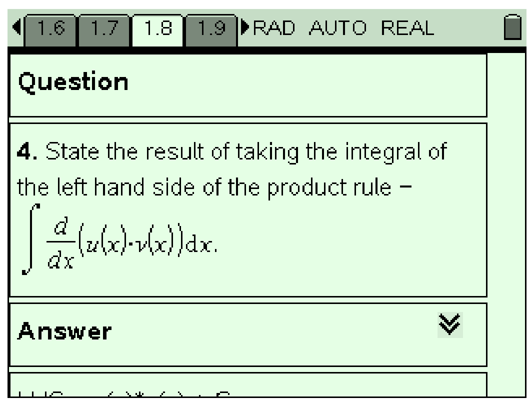
Step 1: Begin with discussion concerning integration methods and review standard integrals. Some review of substitution methods of integration is recommended. Students should be aware that not all functions can be integrated symbolically.



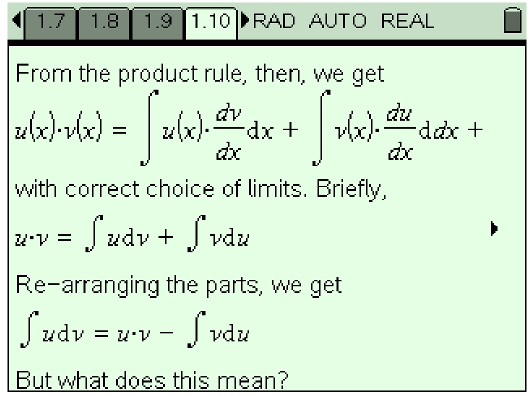
Step 2: Students should be given opportunities to review and consolidate their skills and understandings related to both product rule and the process of integration (particularly drawing distinction between the integral of a function and the area under the curve of the graph of that function).



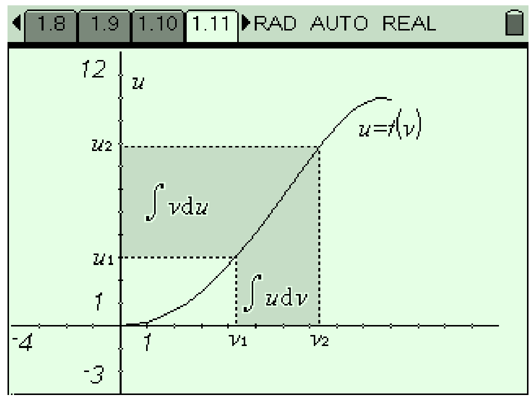
Step 3: The idea of taking the integral of a derivative may well be new to students and should be handled carefully, ensuring that errors and misunderstandings are exposed if possible. Students should be aware that integrals evaluated by CAS generally do not include the constant term.



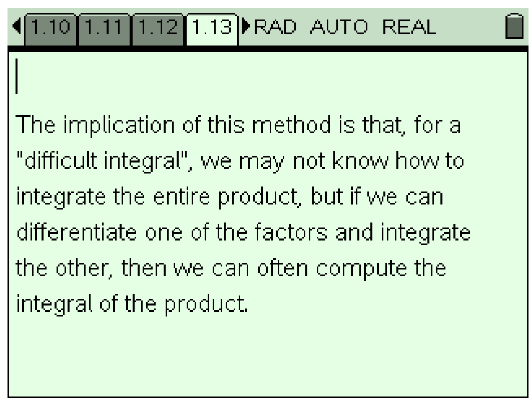
Step 4: Integrating the product rule and rearranging the parts provides an expression which proves to be the basis for our method of integration by parts. However, it is recommended that this process is not simply assumed as valid, but that students be given the opportunity to engage with it, to discuss and to challenge.



Step 5: A graphical approach can help students to appreciate the significance of each of the component parts of the integration by parts statement: the areas under the curve **with respect to u** and **with respect to v** are key concepts here. Interestingly, it is the product, ***u*v*** which is most difficult to draw from this scenario. It represents a shorthand for the difference product, $u_2 * v_2 - u_1 * v_1$. This is shown on the diagram as the L-shaped area between the curve and the axes, between the given limits.



Step 6 Drawing together these observations students should be assisted to see the significance of this method, which involves correctly re-stating the function in question in order to take the derivative of one component and the integral of the other



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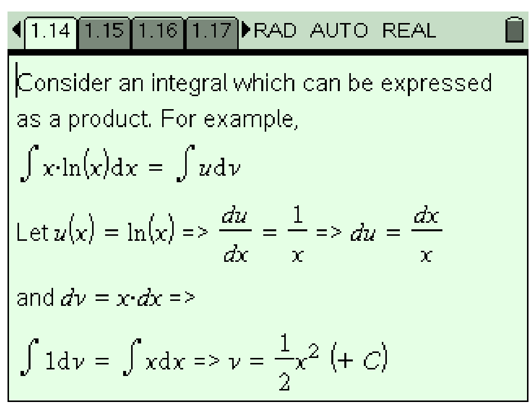
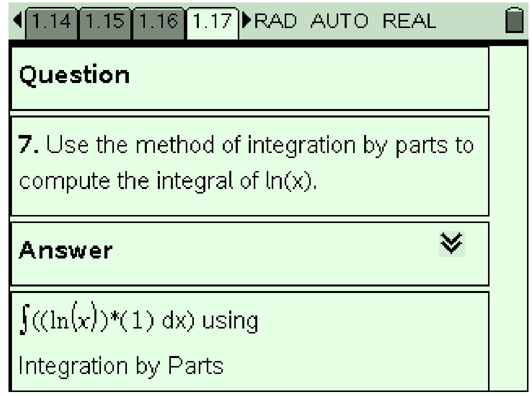
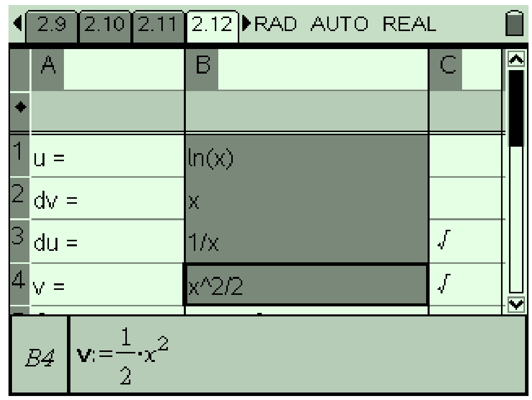
Step 7 Several worked examples are provided, beginning with the relatively straight-forward ***x*ln(x)***. The challenge soon emerges: how to identify which part of the product becomes ***u(x)*** and which part becomes ***dv***. Whether this choice actually makes any difference to the result becomes the basis for an extension exercise later.

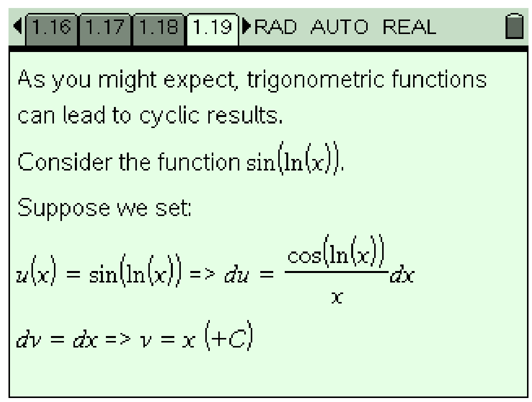


Step 8 The ability to evaluate certain harder integrals which do NOT appear to be products provides the basis for the next key worked example. In the case of the natural logarithm, we do not at present know the integral for this function, but we do know its derivative, and can easily integrate 1! This opens the doors as an approach for a wide range of challenging integration questions.

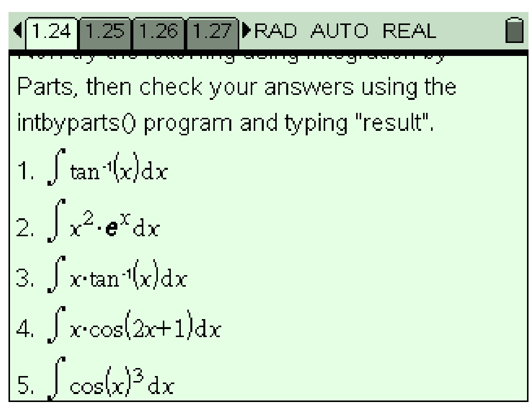




Step 9 As with the preceding CAS-based Calculus activities, a prepared program, in this case, **intbyparts(fn1, fn2)**, is available for students to verify their own work, check their conjectures and results, and to serve as a model for a complete worked solution to such questions. Note that the two input components are derived simply from the given function – the first is defined as ***u(x)*** and the second as ***dv/dx***. The final step in the program is left to the students, who can check their answer by typing the variable **result**.



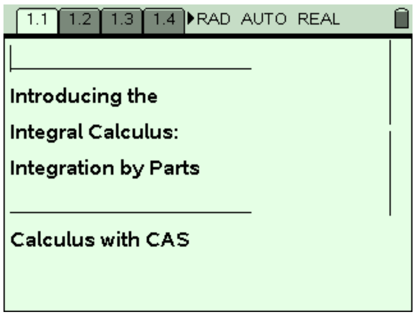
Step 10 The third worked example involves trigonometric functions, which often lead to cyclic results – an integral involving a ***sin*** will generally lead to another integral involving ***cos***. Applying the rule a second time will often result in another ***sin*** result, which may be then set as an equation involving the original integral and solved accordingly. Reduction rules may also be developed in this way, leading to a method for integrating trig functions of higher powers.

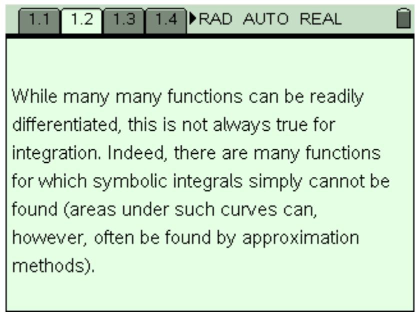


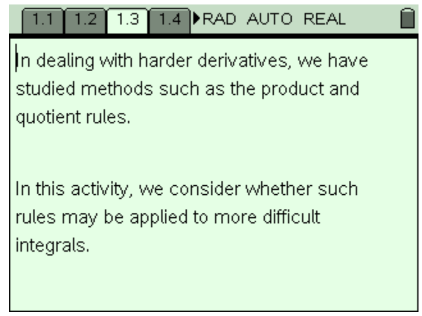
Step 11 Practice questions are provided and then two extension challenges: does the order of choice make any difference (sometimes), and is there an integration equivalent to the quotient rule (no, too difficult to establish).

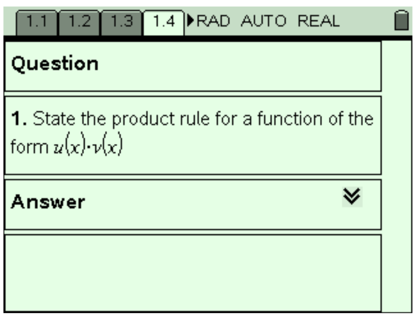
Introducing the Integral Calculus: Integration by Parts – ID: XXXX

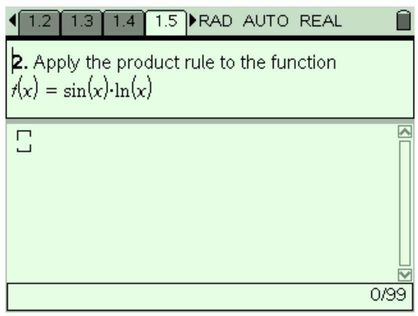
(Student)TI-Nspire File: *CalcActXX_Integration_by_Parts_EN.tns*

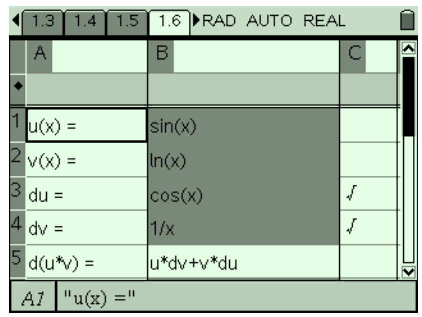


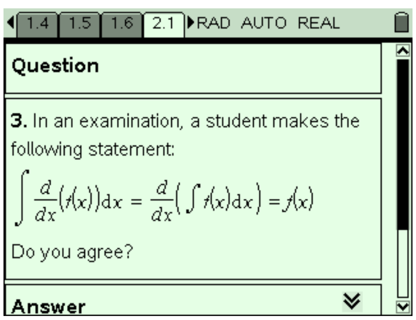


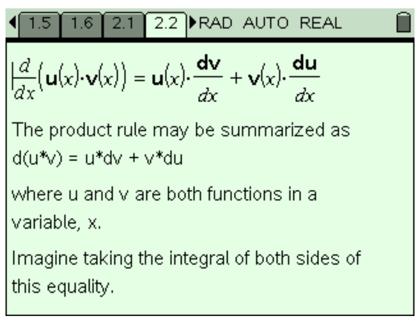


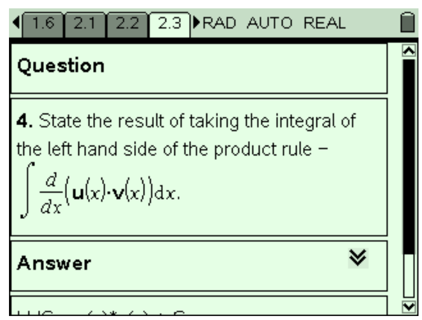


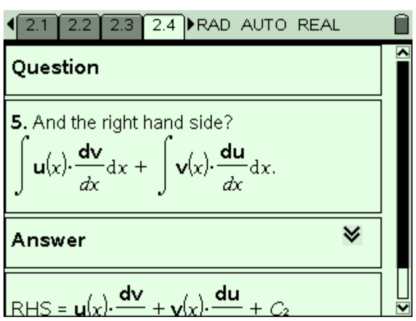


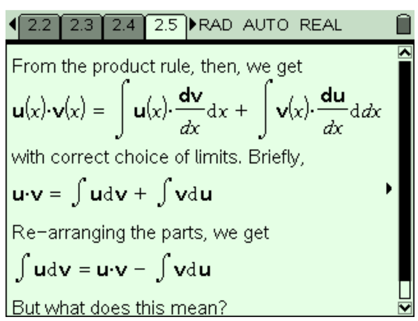


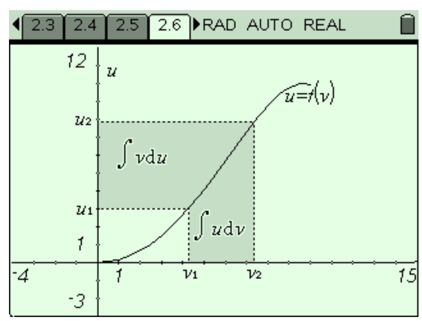












2.4 2.5 2.6 2.7 RAD AUTO REAL

Question

6. Carefully explain the relationship between the areas shown on the graph and the relationship derived:

$$\int_{v_1}^{v_2} u\,dv = u\cdot v - \int_{u_1}^{u_2} v\,du$$

Answer

2.5 2.6 2.7 2.8 RAD AUTO REAL

The implication of this method is that, for a "difficult integral", we may not know how to integrate the entire product, but if we can differentiate one of the factors and integrate the other, then we can often compute the integral of the product.

2.6 2.7 2.8 2.9 RAD AUTO REAL

Consider an integral which can be expressed as a product. For example,

$\int x\cdot\ln(x)dx = \int \mathbf{u}\,d\mathbf{v}$

Let $\mathbf{u}(x) = \ln(x) \Rightarrow \frac{d\mathbf{u}}{dx} = \frac{1}{x} \Rightarrow d\mathbf{u} = \frac{dx}{x}$

and $d\mathbf{v} = x\cdot dx \Rightarrow$

$\int 1 d\mathbf{v} = \int x dx \Rightarrow \mathbf{v} = \frac{1}{2}x^2\ (+C)$

2.7 2.8 2.9 2.10 RAD AUTO REAL

Integration by parts gives:

$\int u\,dv = u\cdot v - \int v\,du$

$\int \ln(x)\cdot x dx = \ln(x)\cdot\left(\frac{1}{2}\cdot x^2\right) - \int\left(\frac{1}{2}\cdot x^2\right)\cdot\frac{1}{x}dx$

$\int \ln(x)\cdot x dx = \frac{x^2\cdot\ln(x)}{2} - \frac{x^2}{4} + C$

2.9 2.10 2.11 2.12 RAD AUTO REAL

Try this example using the spreadsheet on page 2.12, and then check your results using the program **diff_quotient(u(x),v(x))** which gives the step-by-step process, and generates the final answer as the variable, **result**.

2.9 2.10 2.11 2.12 RAD AUTO REAL

	A	B	C
◆			
1	u =	ln(x)	
2	dv =	x	
3	du =	1/x	√
4	v =	x^2/2	√
5	∫u.dv =	u*v − ∫v.du	

A1 "u ="

2.10 2.11 2.12 2.13 RAD AUTO REAL

∫((ln(x))*(x) dx) using

Integration by Parts

∫(u*dv) = u*v − ∫v*du

u(x) = ln(x)

du = $\frac{1}{x}$ dx

dv = x dx

1/1

2.11 2.12 2.13 2.14 RAD AUTO REAL

This method can even be used for functions for which we do not have a closed form integral, for example, $\ln(x)$.

Treat the function as a product with 1:

$\int \ln(x).1dx \Rightarrow \mathbf{u} = \ln(x)$ and $d\mathbf{v} = dx$.

2.12 2.13 2.14 2.15 RAD AUTO REAL

Question

7. Use the method of integration by parts to compute the integral of ln(x).

Answer

∫((ln(x))*(1) dx) using

Integration by Parts

2.13 2.14 2.15 2.16 RAD AUTO REAL

Check your previous result by running the program **intbyparts(ln(x),1)** and then typing "**result**".

∫((ln(x))*(1) dx) using

Integration by Parts

∫(u*dv) = u*v − ∫v*du

u(x) = ln(x)

1/1

2.14 2.15 2.16 2.17 RAD AUTO REAL

As you might expect, trigonometric functions can lead to cyclic results.

Consider the function $\sin(\ln(x))$.

Suppose we set:

$\mathbf{u}(x) = \sin(\ln(x)) \Rightarrow d\mathbf{u} = \frac{\cos(\ln(x))}{x}dx$

$d\mathbf{v} = dx \Rightarrow \mathbf{v} = x\ (+C)$

2.15 2.16 2.17 2.18 RAD AUTO REAL

$\int \sin(\ln(x)).1dx$

$= x\cdot\sin(\ln(x)) - \int x\cdot\frac{\cos(\ln(x))}{x}dx\ (+C)$

$= x\cdot\sin(\ln(x)) - \int \cos(\ln(x))dx\ (+C)$

which sends us back to find the integral of $\cos(\ln(x))$.

2.16 2.17 2.18 2.19 RAD AUTO REAL

Question

8. Find $\int \cos(\ln(x))dx$

Answer

$\int \cos(\ln(x))dx$

$= x\cdot\cos(\ln(x)) + \int \sin(\ln(x))dx$

2.17 2.18 2.19 2.20 RAD AUTO REAL

Question

9. Substitute the previous result for $\cos(\ln(x))$ into the integration by parts result for $\sin(\ln(x))$.

Answer

$\int \sin(\ln(x)).1dx =$

2.18 2.19 2.20 2.21 RAD AUTO REAL

intbyparts(sin(ln(x)),1)

∫((sin(ln(x)))*(1) dx) using

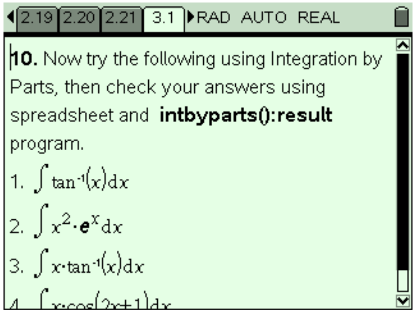
Integration by Parts

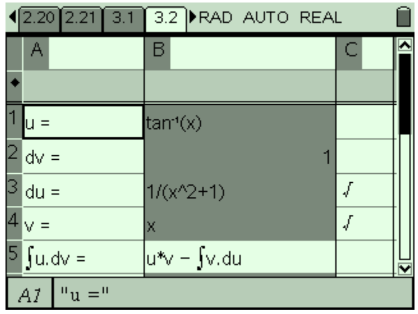
∫(u*dv) = u*v − ∫v*du

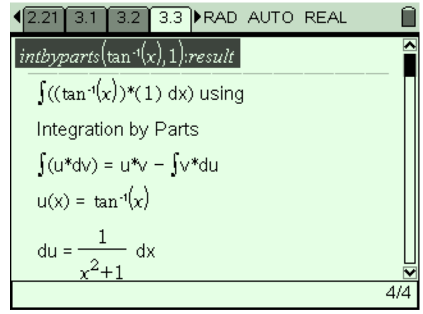
u(x) = sin(ln(x))

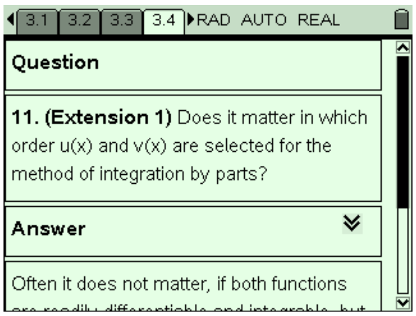
du = $\frac{\cos(\ln(x))}{x}$ dx

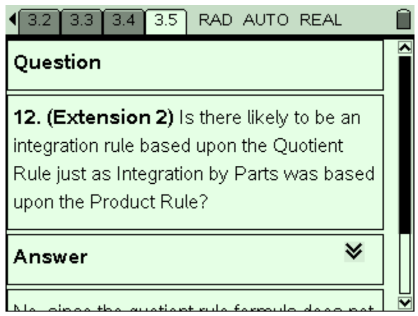
2/2


2.19 2.20 2.21 3.1 RAD AUTO REAL
10. Now try the following using Integration by Parts, then check your answers using spreadsheet and intbyparts():result program.
1. ∫ tan⁻¹(x)dx
2. ∫ x²·eˣdx
3. ∫ x·tan⁻¹(x)dx


2.20 2.21 3.1 3.2 RAD AUTO REAL
A
B
C
u =
tan⁻¹(x)
dv =
1
du =
1/(x^2+1)
v =
x
∫u.dv =
u*v − ∫v.du
A1 "u ="


2.21 3.1 3.2 3.3 RAD AUTO REAL
intbyparts(tan⁻¹(x),1):result
∫((tan⁻¹(x))*(1) dx) using
Integration by Parts
∫(u*dv) = u*v − ∫v*du
u(x) = tan⁻¹(x)
du = 1/(x²+1) dx
4/4


3.1 3.2 3.3 3.4 RAD AUTO REAL
Question
11. (Extension 1) Does it matter in which order u(x) and v(x) are selected for the method of integration by parts?
Answer
Often it does not matter, if both functions


3.2 3.3 3.4 3.5 RAD AUTO REAL
Question
12. (Extension 2) Is there likely to be an integration rule based upon the Quotient Rule just as Integration by Parts was based upon the Product Rule?
Answer