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Grade level: secondary (Years 10-12)
Subject: mathematics
Time required: 90 minutes

Activity overview

Many special properties can be shown to be associated with cubic functions. This activity investigates tangents to the cubic function and their relationship with the roots of the function. The investigation involves functions, solving for roots, calculus and graphing techniques.

Background

Some students have difficulty in interpreting problems presented and analysed using algebraic techniques. The use of dynamic graphing technology allow these students to access problems in a more visual manner and relates the algebra with the visual image.

Concepts

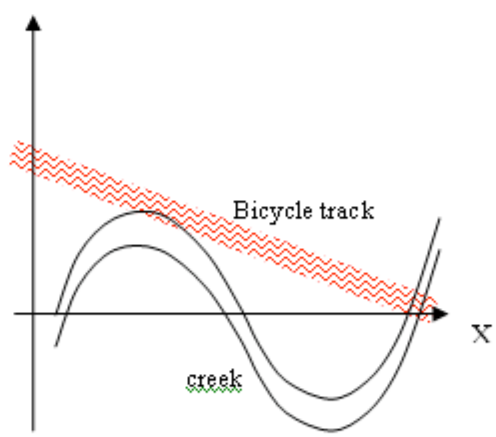
Function notation, root values, gradient functions, tangents, coordinates

Teacher preparation and Classroom management tips

Set the problem that allows students to "picture" the situation in real practical terms. This also engages the student as they see a "use" for the mathematics involved in the task.

It is useful for students to free draw a cubic function with 3 root values either on a large sheet of paper, our outside using chalk or in the sand. Allow the students to explore how a straight edge (e.g. metre rule or taut string) can pass through a root and also be a tangent to the curve. Can they estimate where the straight edge must touch the curve to form the tangent? Allow the groups to compare their drawings and hence form a conjecture that can be tested using *TI-Nspire*.

A practical situation may involve a bicycle track being built near a creek. The track needs to be linear and go from a bend on the creek to a bridge further along the creek.



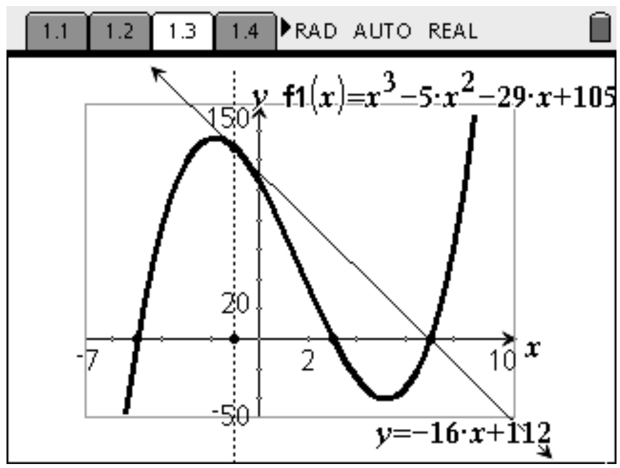
Technical prerequisites

With *TI-Nspire*, students should be able to:

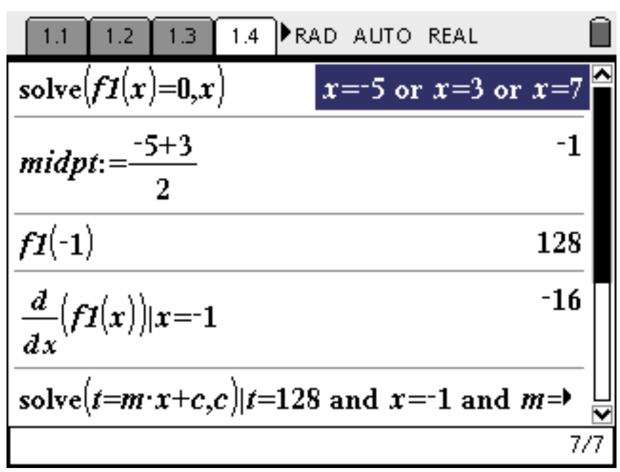
- graph functions
- solve systems of equations, including those with parameters (including the intersection of graphs)
- use calculus tools (or other appropriate methods) to find gradients
- construct sliders to alter parameters.

Step-by-Step Instructions

Show that in the given case of a cubic function (with three distinct roots) the tangent drawn at the midpoint of the first two roots passes through the third root. Refer to graph to visualise the problem if the pre-task introduction was not used.



Investigate the specific function
$f(x) = x^3 - 5x^2 - 29x + 105$
Show that it has 3 distinct roots.



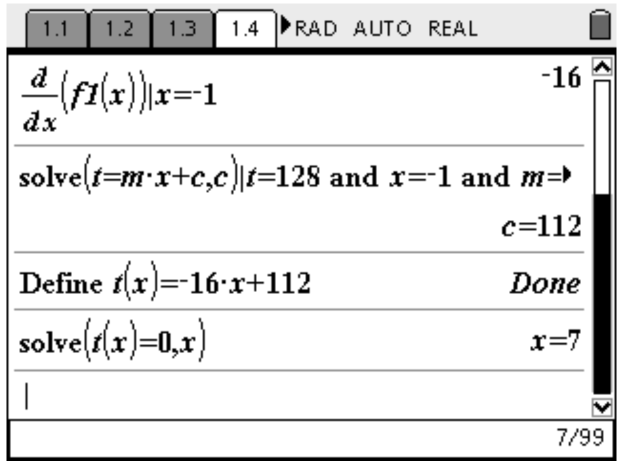
(**TI-Nspire CAS screen shown)**

Find the tangent at the midpoint of the first two roots.

Determine coordinates of the midpoint by finding the *x*-value first.

Find the gradient at the midpoint.

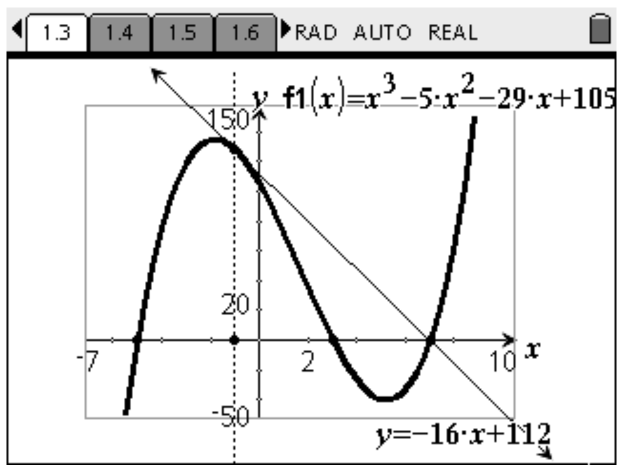
Let the rule of the tangent be $t = m*x + c$



(TI-Nspire CAS screen shown)

Show algebraically and graphically that the tangent crosses the *x*-axis at the third root value.

In the Graphs & Geometry application type in the functions *f*(*x*) and *t*(*x*) in the Graph Box editor.



Generalise the problem algebraically using CAS

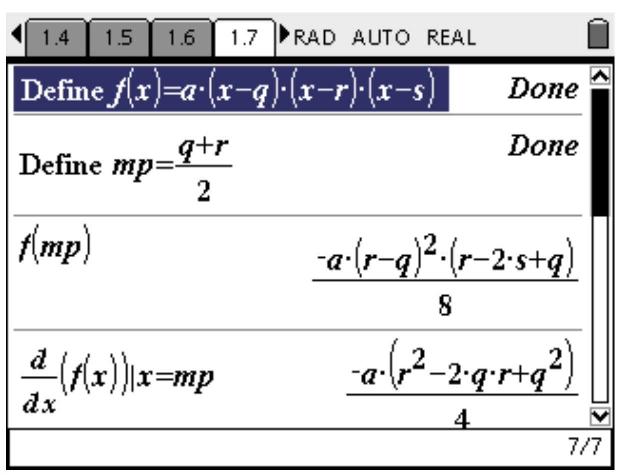
Define *f*(*x*). (remember to insert *)

Let the midpoint abscissa of the first two roots be *u.*

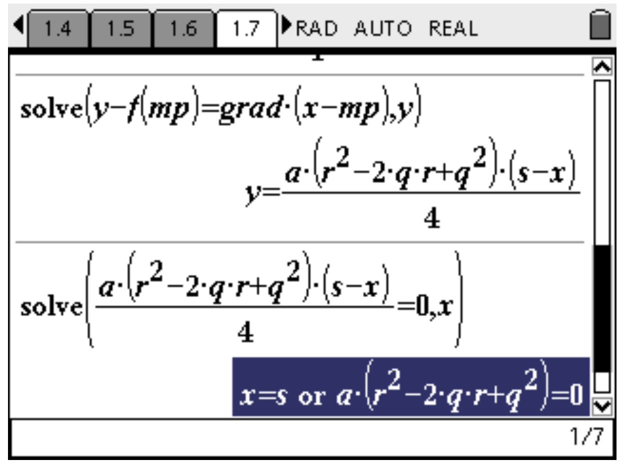
Find *f*(*u*)

Using a calculus method find the gradient, m, of the curve, and hence the tangent, at $x = u$.

Using $(y-y_1)=m(x-x_1)$, or otherwise, find the equation of the tangent.



Solve for *x* to determine if the third root value, *s*, of *f*(*x*) is a solution of *t*(*x*)

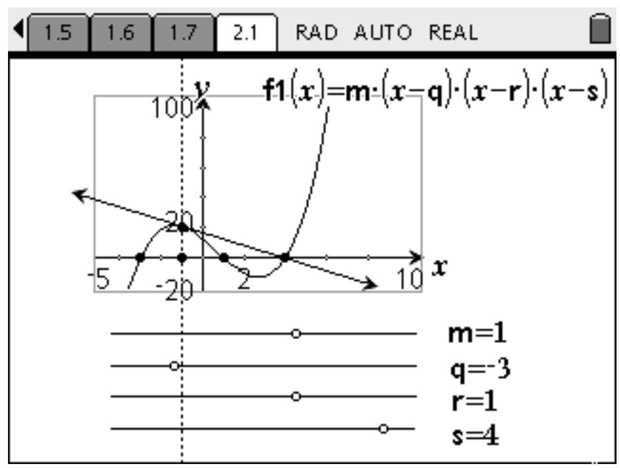


Solve graphically

Create sliders to dynamically change the parameters of *f*(*x*) and *t*(*x*).

a. Select **MENU > Points & Lines > Segment** and draw this at the bottom of the page from left to right.

b. Select **MENU > Points & Lines > Point** and place a point-on the segment.

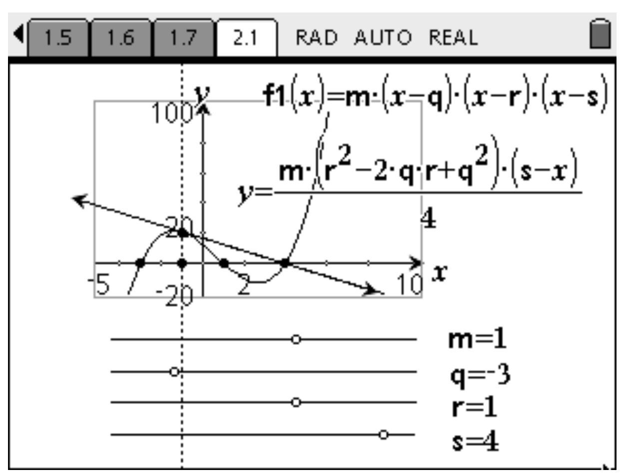


c. Select **MENU > Measurement > Length** and place the cursor on the left-hand point on the segment, click, then place the cursor on the other point and click. You have now obtained the length between the points.

d. When you grab and move the added point on the slider the *"length value*" will alter. This will be used to change the *a*-value in the following steps. *If you wish you can store this measurement value as the parameter a. Select **VAR** and drag cursor over the value and click. Select **Store Var** and move cursor to the left and type a = and press Enter.*

e. Create the other 3 sliders. (i.e. *q, r, s*)

f. Select **MENU > Tools > Text** and enter $f(x) = a*(x-q)(x-r)(x-s)$ on the page. Press Enter. Select **MENU > Tools > Calculate** and place the cursor over the function expression. Left click and move the cursor away. The yellow box asks for *a*. Move cursor on to the *length value* you determined to represent *a*, left click. Repeat for the other parameters and then place the cursor on the *x*-axis and click. You have now drawn in the function.



g. Repeat the above for *t*(*x*).

h. Move the added point/s on the slider/s and see what happens. Make sure that you have a long enough *y*-axis.

Assessment and evaluation

There are numerous options for evaluating the success of the lesson.

Discussion after initial exploration phases using paper and rule and subsequent

A written report on the findings with relevant screen dumps would be appropriate.

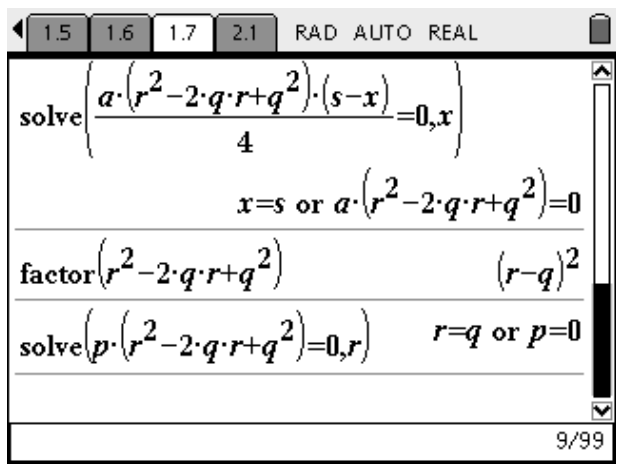
Under structured supervision students could be presented with another specific example, perhaps a negative cubic as a new situation and asked to show the conjecture still applies to this example.

CAS Activity extensions

The general solution suggests that the individual roots are not important. i.e. solving shows $x = s$ or q or r.

What does this mean in the context of the problem?

Do you actually need 3 distinct roots or will it work for 2 distinct roots (1 repeated root)? Will it work for 1 distinct root. i.e. point of inflection?



Set $a = 1$ and test your conjecture. Graphically explore the 1 and 2 distinct roots situation using the sliders.

Find whether the relationship found in this task is still relevant for cubic functions when $a < 0$.