TI-nspire™

Looking into Parabolas

By Peter Fox
Grade level: secondary (Years 11-12)
Subject: mathematics
Time required: 90 minutes

Activity overview

A parabolic reflector can be studied visually using a light box found in many science departments. Light rays parallel to the principle axis reflect back through the focal point. Can mathematics be used to model this situation and subsequently determine the location of such a point?

Background

Focus and directrix properties are generally studied jointly and incorporate a study of eccentricity. This activity is aimed at using the physical characteristics of a parabolic reflector to determine the location of the focal point without the necessity to introduce the directrix or eccentricity*. TI-Nspire CAS* provides an environment where students can use their understanding of reflection to explore a mathematical solution incorporating calculus.

Concepts

Reflection, Geometric construction, Co-ordinate Geometry, Algebraic Modeling, Function Graphing, and differential calculus

Teacher preparation

The practical and conceptual nature of this activity makes the integration of a physical model very important. Students need to see how light is reflected from a flat surface, the angle of incidence is equal to angle of reflection. The next step is to introduce a parabolic reflector. A single light ray, reflecting from the surface of the parabolic reflector incorporates an understanding of the tangent to the curve. Light behaves in the same way as it does for the flat mirror, the angle of incidence is equal to the angle of reflection; these angles are measured from the tangent to the curve. Some time should be spent discussing this aspect as the tangent concept helps students make the connection with calculus in their solution process. Next, the students need to see how a series of light rays, parallel to the principle axis, are reflected back through a single point, the focal point. This component of the demonstration is the essence of the problem to be solved. Student calculations can be simplified if they identify the situation where the incident ray makes an angle of 45° with the tangent. At the completion of the demonstrations, students should have a clear picture of the concepts which will help them in a discussion of the solution process.

Classroom management tips

Whether a teacher leads the practical demonstration or students perform them on an individual basis is up to the individual. This practical application spotlights the concept of tangent. Important student discoveries should be listed on the board through teacher lead discussion. Key terms such as reflection, tangent and comments about the angle of reflection and the angle of incidence are all important contributions. Students need to connect the physical observations with the mathematical tools that can be used to formulate their solution.

Prior to working with the technology, students should draw diagrams. Students should draw tangent lines to points on the parabolic reflector with incident light rays reflecting through a single point. This helps provide the necessary scaffolding, drawn from their practical experiences, to help solve the problem in a mathematical context. Mathematically, students need to be able to determine the derivative of a basic polynomial, solve linear and systems of linear equations and substitute values into an expression.

Technical prerequisites

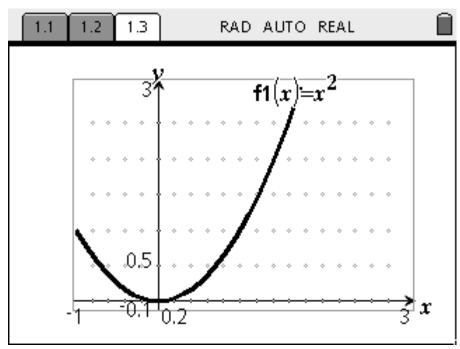
Students should know how to:

- Graph a parabola and manipulate it using the selection arrow. (Graphs & Geometry application)
- Place a point on a curve and draw a tangent to the point on this curve. (Graphs & Geometry application)
- Draw parallel and perpendicular lines. (Graphs & Geometry application)
- Reflect a line in another line.
- Define and operate upon expressions as preparation for solving relevant equations (using CAS if available) (Calculator Tool)

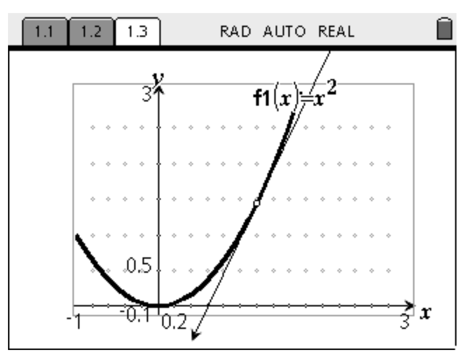
Step-by-step directions

Stage 1: Understanding the task and preliminary exploration.

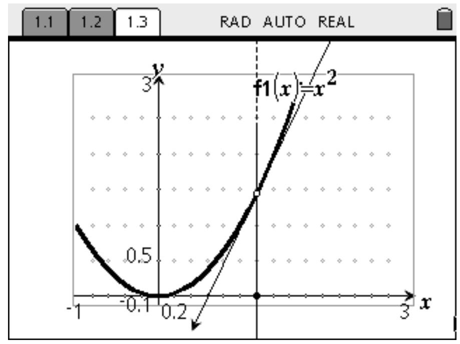
a) Using the Graphs & Geometry application, graph the function $y = x^2$.



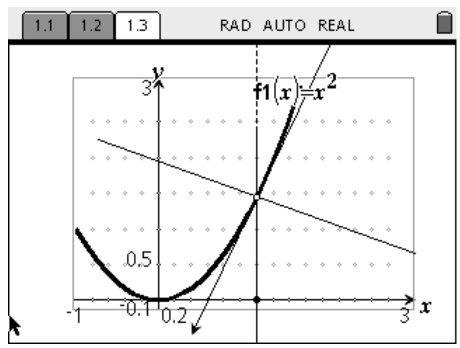
b) Place a point on the graph and draw a tangent to this point.



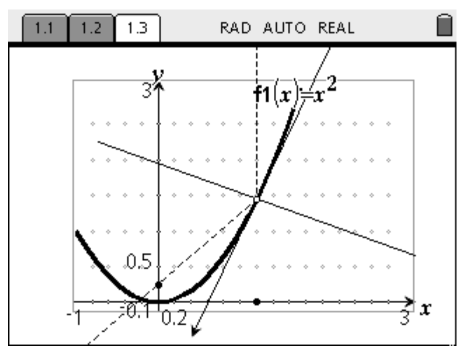
c) Construct a line perpendicular to the x axis, passing through the point on the curve.



d) Construct a line perpendicular to the tangent, passing through the point on the curve. This line is referred to as the ‘normal’. This line will be used to reflect the incident ray.



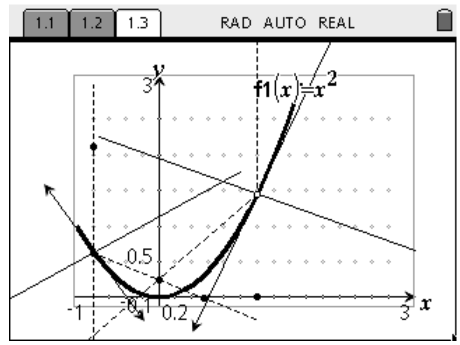
e) Place a point on the vertical line. Reflect the point through the normal and construct a ray from the point on the curve through the reflected point. The vertical line represents the ‘incident light ray’. The reflected line is the ‘reflected light ray’.
Note:
It is possible to reflect the incident line; however, this creates unnecessary complication in the diagram.



f) Use the pointer tool to drag the point on the parabola. Observe what happens to the reflected ray.

g) Repeat the previous steps to create a second reflected ray through a different point. Take it in turns to drag the two points around the parabola. What do you notice about the two reflected rays?

h) Construct a point where the two reflected rays meet and measure the coordinates of the point. Note what happens to the location of the point when the parabola is dilated.



Stage 2: Calculus Investigation.

1. Determining of the coordinates of the focal point can be made very simple by strategically placing the point on the curve where the incident light ray strikes.
   a. What is the gradient of the reflected ray when the incident light ray makes an angle of 45° with the parabola?
   b. Determine the y – coordinate of the focal point.
   c. Determine the x – coordinate of the focal point and explain how this point was determined.
2. To verify the solution obtained in the previous question, the location of the point on the curve is located such that the incident light ray makes an angle of 30° with the parabola.
   a. What gradient(s) are possible for the reflected ray?
   b. What are the equation(s) for the reflected ray(s)?
   c. Determine the location of the focal point and compare the result with the previous answer.
3. The dynamic nature of the software and diagram make visualizing the general solution simple. Algebra and calculus can also be used to generalize the solution. For each of the functions, determine the location of the turning point:
   a. $f(x) = ax^2$
   b. $f(x) = a(x-h)^2 + k$
   c. $f(x) = ax^2 + bx + c$

Assessment and evaluation

Contributions and involvement in the practical component, combined with the answers to the questions formulate the major component of the assessment. The learning process for students should not terminate at the assessment stage. Some of the questions in this task can be solved in a number of ways. For example, 2C could be done algebraically using the answer to 2B or it could be done using the same calculus approach used to determine the previous answers. Students should be invited to share their methods and thinking behind their solution process.

Extension

A range of opportunities exist that will extend this task and challenge students thinking. Open ended questions such as: Determine an equation to a parabola with a focal point at (1, 2). The solution to this problem of course is not unique. Students that attempt to solve the problem may find that there is not enough information to solve a system of equations. This should leave the students to realize that multiple solutions exist. A discussion at this point is worthwhile to provide students directions on how to move on and determine at least one solution, and perhaps a general solution, a considerably harder task.

If students are familiar with parametric equations, this task can be taken to a much higher level by providing students with the set of parametric equations:

$$x(t) = \frac{t(t+1)}{11}$$

$$y(t) = \frac{t^2 - 21t + 110}{11}$$

This set of parametric equations produces a 'parabola' that has been rotated through 45°. The principle axis is therefore the line $y = x$. Incident light rays need to be parallel to the principle axis. Reflected rays can be constructed geometrically using the same principles applied in this task. The mathematical solution is significantly more complicated but can be used as a context for other differentiation techniques such as implicit differentiation.