# The Case of the Diminishing Square

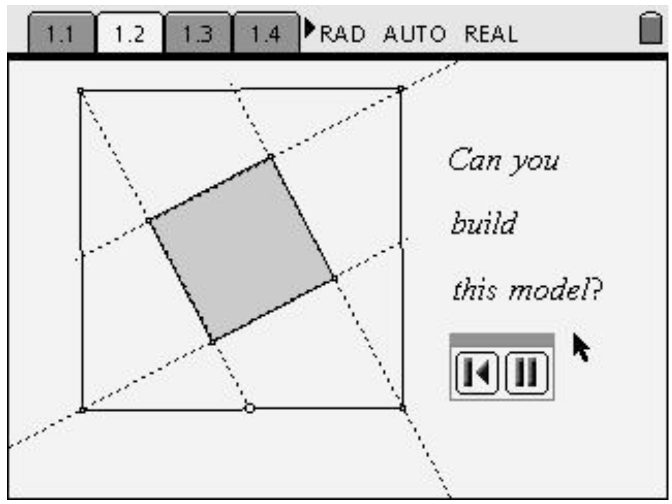
By Stephen Arnold
Grade level: secondary (Years 9-12)
Subject: mathematics
Time required: 90 minutes

## Activity overview

Study the diagram provided. A smaller square has been constructed inside a larger square, as shown.

A point ***x*** is located on the base of the larger square. (As shown) The smaller square is constructed using similar points on each of the remaining sides of the larger square. If ***x*** is the midpoint of the base, what is the ratio between the area of the larger square and the smaller square?

Explore the relationship between the position of this point and the area of the smaller square.




## Background

This problem is enhanced through the use of interactive geometry software, which supports its creation and ease of manipulation. Students need to appreciate the approximate nature of such software and that both a geometric and an algebraic proof may be required. This problem is introduced as a square within a square. For completeness, it may be valuable to have students explore a proof that this construction leads to a square and not a rhombus or other quadrilateral.

## Concepts

Geometric properties of right-angled triangles, Pythagoras' theorem, similarity, algebraic modeling

## Teacher preparation

Prior to attempting this activity, students should be familiar with properties of right-angled triangles and their altitudes. The elegance of this problem lies in the simplicity of its posing, relative ease of construction and the challenge of arriving at an algebraic solution. There are many ways of investigating the relationship. A numerical approach via the interactive geometry environment is suitable for less academic students through to a complete algebraic solution for more able students.

## Classroom management tips

This problem is best set as a collaborative problem-solving task. Students can work in pairs or small groups using the technological tools supplied. Students may also find it useful to create a physical model of the situation in order to gain a clear insight to the symmetry of the construction. Regular breaks when students share their thinking and progress with the class serve as useful scaffolding.

## Technical prerequisites

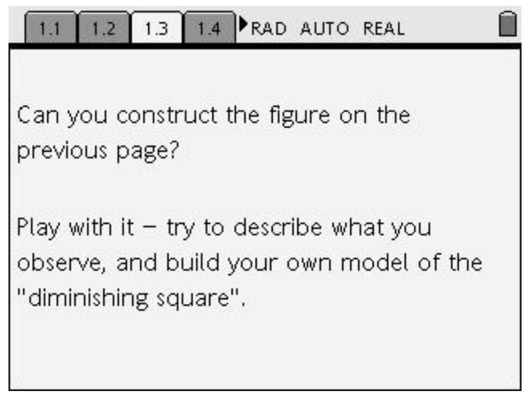
Students should know how to:

- Construct a square using axes;
- Reflect points and shapes in lines;
- Measure length and area of geometric objects;
- Transfer measurements from geometric figure to axes for graphing of relationships.

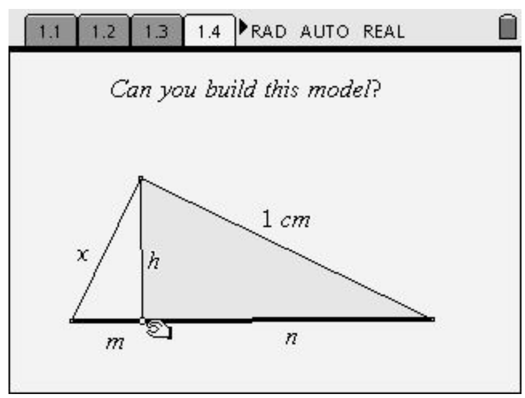
## Step-by-step directions

1. Depending on the level of ability and experience of the class, students may either be expected to build their own geometric model for this task, or be provided with one. (As shown).

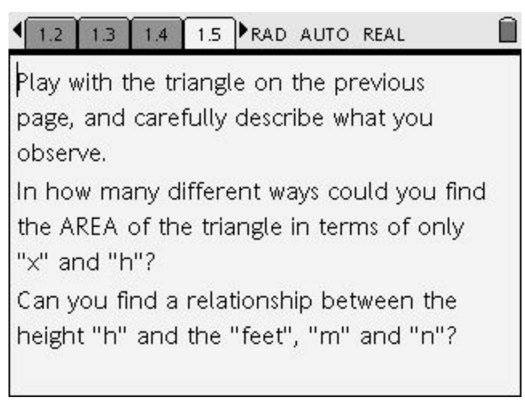
   Ideally the teacher should demonstrate the model by dragging (or animating) the variable point (***x***) along the axis. Allow students to observe the relationship. Students may be challenged to build their own figure.



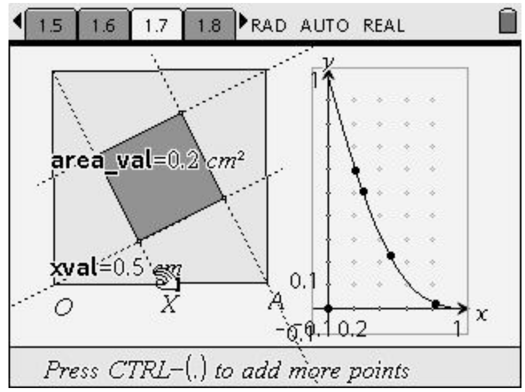

2. Introducing two right-angled triangles as key components of the diminishing square construction provides further scaffolding for students, and lays a basis for the later algebraic approach to the proof.




3. Engaging students with the "diminishing square" task now within a more algebraic context encourages them to explore the model provided physically at first (by moving and observing), then numerically, graphically and finally algebraically. In this way they are provided with the tools they need to investigate the task in a systematic way, to build and test their own conjectures.



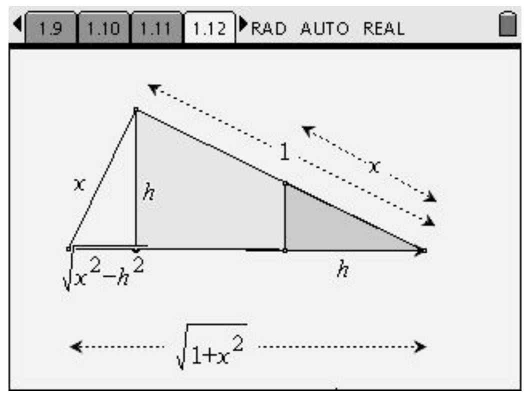
4. Much may be gained by the teacher collecting from students their understanding of the relationship between the x-variable and the area. This data may be entered into lists and graphed, or an intuitive graphical model may be explored.




5. The final stage of the task lies in students building an algebraic model of the relationship between the value of x and the area of the diminishing square.

   In this case, the exploration of an algebraic model of the inverted right-angled triangle shown can be used for a solution. In particular, students need to realize that the side of the small (diminishing) square (**s**) may be found by finding the area of the right-angled triangle in two different ways, and then solving these to eliminate the variable **h**.

   Clearly the opportunity exists here for a possible CAS extension, making use of the additional tools offered on the CAS platform to assist students in building these quite complex algebraic models.



## Assessment and evaluation

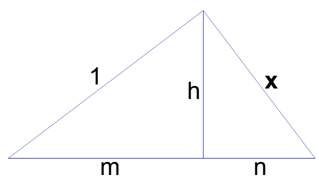
Assessment for this activity should build from the particular to the general. Students initially use the model to answer simple questions and draw connections between various mathematical representations then gradually work towards an algebraic solution which matches their numeric results, but offers the potential for both greater precision, and for prediction (clear opportunities for CAS extension).

1. What is the ratio of the areas of the small square to the large square when x is ½ the length of the large square?

   The area of the smaller square is exactly one fifth of that of the larger square – five such smaller squares would fit within the larger square!

2. Find the value for **x** when area of the small square is half the area of the larger square.

   **X = 0.81**
   Assuming the larger square has a length of one unit. Alternatively, the ratio is 0.81 of the length of the larger square.




3. For each of the following problems, assume the larger square has a side length of 1 unit.

   a. Let **h** represent altitude of the triangle shown above. h is also one of the shorter sides of a right angled triangle with **x** as its hypotenuse.
   Find two expressions for the area of the largest triangle within the large square and hence show that $h = \dfrac{x}{\sqrt{1+x^2}}$

   This area can be expressed as either

   $\dfrac{x}{2}$ **OR** $\dfrac{h}{2}\sqrt{1+x^2}$

   b. Show that **s** (the side length of the smaller square) can be expressed as $s = \sqrt{1+x^2} - h - \sqrt{x^2 - h^2}$

   Given that the side length of the large square is 1, then the hypotenuse of the largest triangle is $\sqrt{1+x^2}$. This length is made up of the side **s**, the length **h**, and the small triangle side length $\sqrt{x^2-h^2}$.

   c. Hence express **s** in terms of **x**.

   $s = \dfrac{1-x}{\sqrt{1+x^2}}$

   d. Finally, give an expression for the area of the small square in terms of x.

   $area = \dfrac{(1-x)^2}{1+x^2}$ (also the ratio of the two areas!)

Student reports of this investigation should be clear and detailed; observations and clear explanations of each representation (physical, geometric, graphical and algebraic) should be required. As always, for algebraic modeling, there should be clear recognition of the assumptions made and the limitations of the modeling process.

## Activity extensions

- Other solution processes may be followed in this problem. For example: How could this problem be investigated using similar triangles. (Incorporates the symmetry of the construction and is also useful for proving that the centre ‘quadrilateral’ really is a square.
- Explore this problem in terms of the relationship between the variable **x** and the length **s** of the side of the small square – how does this differ from the use of areas and ratios of areas? What are the strengths and limitations of these approaches?
- Further explore the interesting properties of the inverted right triangle – why is it best constructed using a circle? Did you know that there are three geometric means hidden in this construction?