Playing Rugby

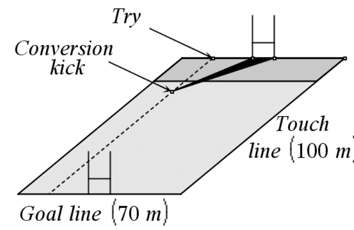
By Philippe Fortin

Grade level: secondary (Years 11-12)
Subject: mathematics
Time required: 90 minutes

Activity overview

What is the optimal position for a rugby player attempting a conversion?

He can freely choose his position along a line, perpendicular to the goal line, passing through the point where the try was scored.



He does need to avoid placing himself too close to the goal line, as this would mean that his kick could be countered by an opposing player, and would also make it more difficult for him to place his ball over the horizontal bar.

The angle θ at which the two posts may be seen will be smaller or greater depending on the position he chooses. It is obviously in his interest to ensure that this angle is as wide as possible.

The aim of this activity is to determine this optimal position and then to study the variations in the angle θ obtained depending on the position where the try was scored.

This task enables us to evaluate the advantage gained by positioning oneself as close as possible to the posts before scoring the try.

A meeting at the top…

Several years ago, a group of French Rugby team officials and a group of teachers preparing for a geometry seminar happened to find themselves sitting around the same table. This took place in the South West of France, an area where this sport is especially popular.

Over this meal, the conversation turned to the ideal position from which to aim the conversion of a try scored close to the base line. The rugby experts were able to confirm to us that the position obtained via the theoretical calculation described in this activity does indeed correspond to the position chosen instinctively by experienced kickers, in the absence of other constraints due to specific circumstances, such as wind force!

Concepts

To carry out this activity, the students will need to be familiar with the classic trigonometric relationships in a right-angled triangle. The direct construction of the curve giving the angle, which depends on the position of the player, requires the use of the inverse tangent function, but this construction is not essential *(it simply enables there to be a further check on the model used).*

The theoretical study of the maximum uses the calculus tangent formula for the difference between two angles. This leads back to a study of the extrema of a rational function.

Teacher Preparation

Before starting this task, it is preferable to ensure that the students have the necessary knowledge. It may therefore be appropriate to carry out the task after a revision of the principal trigonometric relationships, and the properties of the tangent function.

In particular, this task uses the fact that this function is increasing on the interval $[0,\pi/2]$.

Classroom management tips

Students might work in pairs or small groups on the initial examination of the problem. This stage is useful for brain-storming strategies and teasing out implications. It is also a good idea for the teacher to invite some open discussion at intervals throughout the activity, providing further support for those that need it. This is a challenging task, and various degrees of scaffolding will be required for different groups of students. Allowing collaboration is an ideal means of supporting students without too much teacher interference!

Technical prerequisites

To successfully complete this activity, students should know how to:

- Use the Length and Angle functions for measurements of lengths and angles in the **Graphs & Geometry** application.
  Use the Measurement Transfer and Locus tools to create a geometric construction.
- Construct a curve with numerical parameters with the Calculate tool, using various different numerical values and the axis.
- Define a function in **Calculator**, to include working from conditional instructions, calculating a derivative, root-finding.
- Use the **Lists & Spreadsheet** application, including in particular the use of relative and absolute references.
- Set up a link between the variables used in different applications (optional, but desirable).

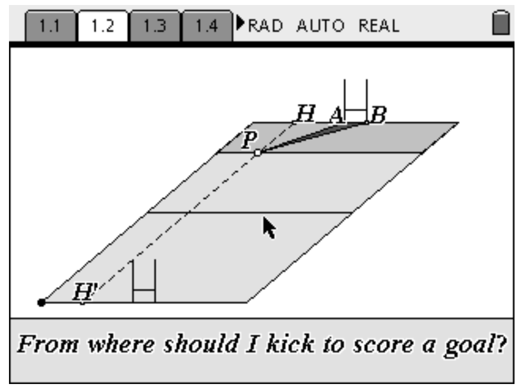
Step-by-step directions

Part one of the activity : observation with the aid of the diagrams provided in the student handout

1. First of all, start by observing the situation described in this task with the aid of the diagram appearing in the student file.

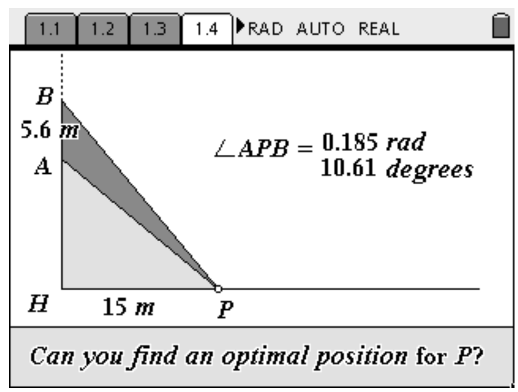
   Point P may be moved along the permitted line.

   The shaded area shows the area which should be avoided as it is too close to the goal line.



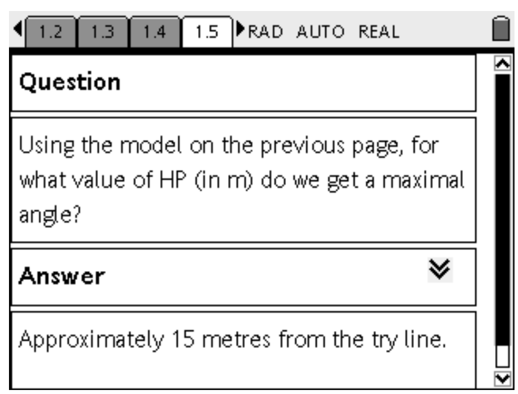
2. The relevant part of the construction appears again in the diagram shown.

   Point P may be moved, and the variation in angle APB observed.



3. By using the above diagram, it is possible to determine by experiment the approximate maximal value of angle APB as shown in the illustration on page 2 (try scored at a distance of 12m from one of the two posts).

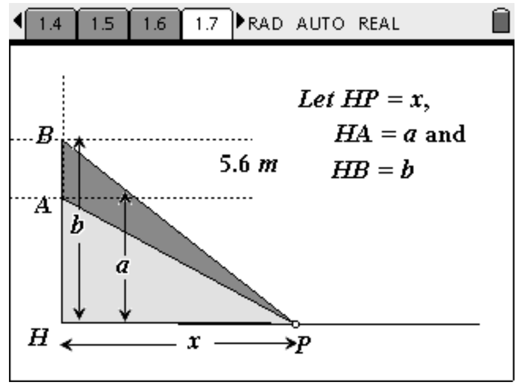
   A Question/Answer model, available in the Notes application, may be used here.



Part two of the activity: finding the optimal distance in the case of a try placed outside the goal posts.

1. The next task is to express the value of angle APB according to the value of the different lengths a=HA, b=HB and x=HP.

   It will be possible in the next part of this activity to verify graphically the validity of this expression.

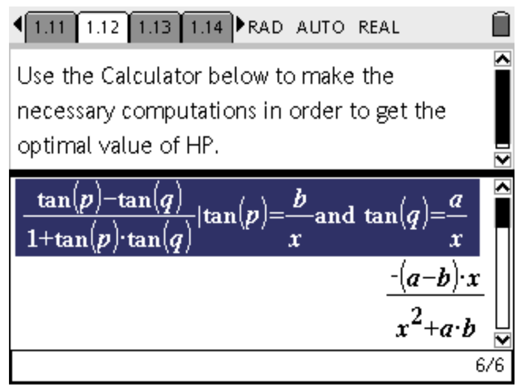


2. Construction of the curve representing the variation in the value of the angle according to the variable x=HP.

☞ This can be carried out directly with the aid of the **Measurement Transfer**, **Perpendicular Line**, **Point of Intersection** and **Locus** tools.

☞

The length HP and the size of the angle on the x-axis and y-axis should be input, which gives the point with these co-ordinates as an intersection of two perpendiculars. The next thing to do is to find out the location of this point when P is varied.

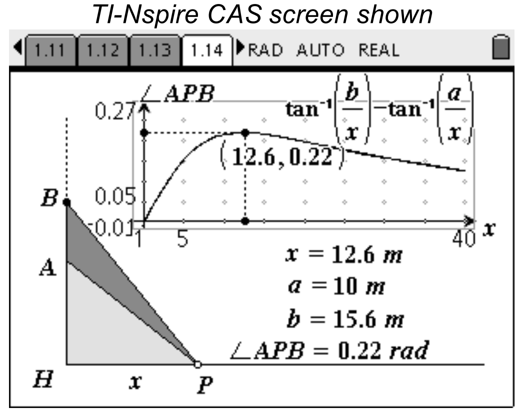


TI-Nspire CAS screen shown

3. It is then possible to check whether the curve obtained corresponds to that obtained by means of symbolic expression.

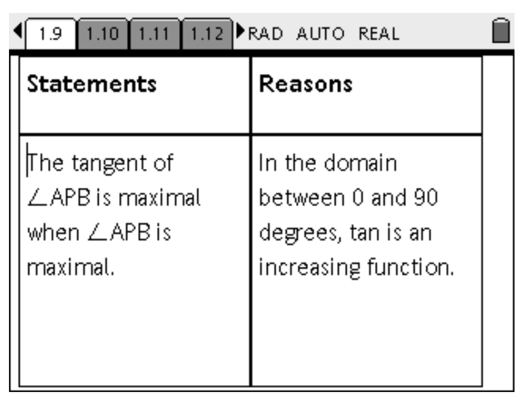
   To do this, the formula should be entered and then the Calculate tool used to obtain the graphic representation of this expression.

☞ Click on axis x when the software asks for the value of x, and on the length measurements of HA and HB to give the values of a and b.



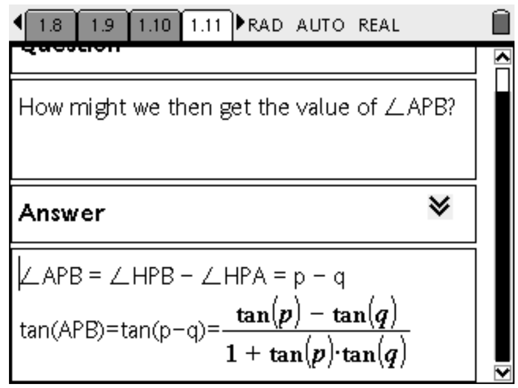
4. These instructions are designed for secondary school classes, where students are not yet able to derive the inverse tangent function.

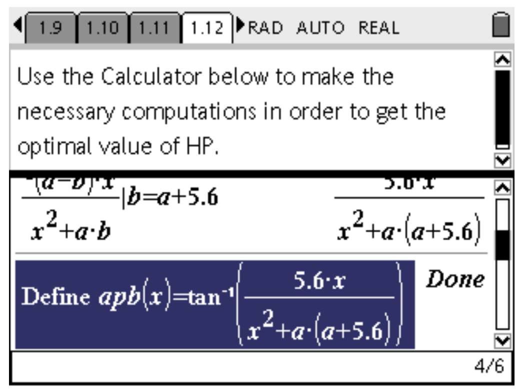
   In order to remain within the scope of this activity, it is preferable to use the rate of change of the tangent function, and to seek the maximum of the function giving the tangent of the angle in question. This allows you to refer back to the study of a rational function.
   A page containing a Statements/Reasons model may be used to request justification for the method used.

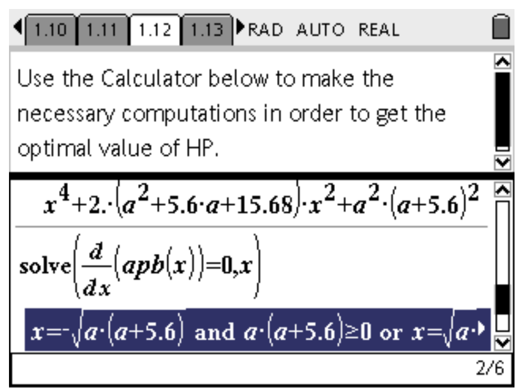


If TI-Nspire CAS is available, then the tangent of the angle can then be calculated directly in the **Calculator** application using the *fExpand* function.

Finally, the correlation that has been obtained may be used to define a function enabling the value of the tangent with respect to *x* to be calculated directly using Define.







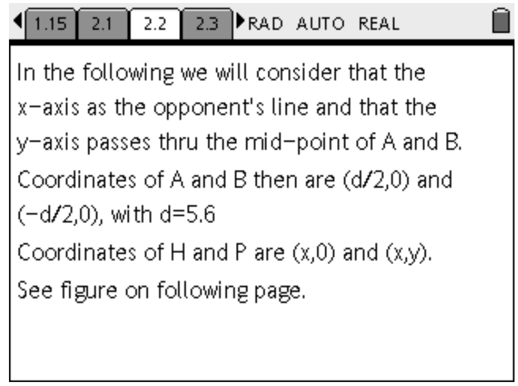
5. Next, using CAS, students can calculate the derivative of this function, and then look for the points where this derivative equals 0, which enables us to show that the optimal position is defined as HP^2=HA.HB i.e. $x^2 = a*b$. Alternatively, the numerical function maximum command will provide this value.

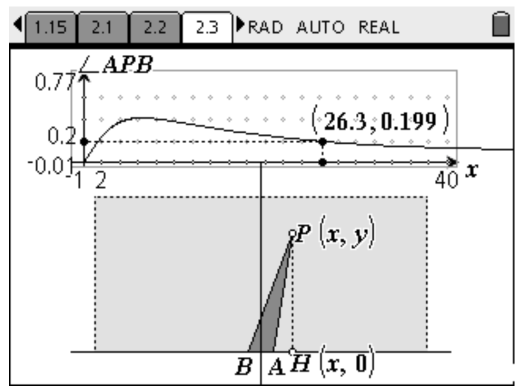
6. It is then possible to calculate this expression numerically for the values HA and HB shown in the original construction.

7. The next task is to make a construction of all of the optimal points.

   This construction may be obtained by using the Length, Measurement transfer, Calculate and Locus tools on the geometry screen.

- This more complete model now allows students to vary the positions of both H and P, and so build a picture of the effects of grounding the ball at different distances from the goal posts, and then taking the kick from various points along that line.





Answers to the questions appearing on the various pages of the student worksheet…

Part 1

1. Optimal distance when HA = 12 m — Approximately 15 m

Part 2

2. Tangent of angle HPA. — $\frac{HA}{HP}$
3. Tangent of angle HPB. — $\frac{HB}{HP}$
4. Calculation of angle APB. — $\tan^{-1}\left(\frac{HB}{HP}\right) - \tan^{-1}\left(\frac{HA}{HP}\right) = \tan^{-1}\left(\frac{b}{x}\right) - \tan^{-1}\left(\frac{a}{x}\right)$
5. Calculation of the tangent of the difference between two angles.

$$f(x) = \tan(\beta - \alpha) = \frac{\sin(\beta - \alpha)}{\cos(\beta - \alpha)} = \frac{\sin\beta\cos\alpha - \cos\beta\sin\alpha}{\cos\beta\cos\alpha + \sin\beta\sin\alpha}$$

$$f(x) = \frac{\dfrac{\sin\beta\cos\alpha}{\cos\beta\cos\alpha} - \dfrac{\cos\beta\sin\alpha}{\cos\beta\cos\alpha}}{\dfrac{\cos\beta\cos\alpha}{\cos\beta\cos\alpha} + \dfrac{\sin\beta\sin\alpha}{\cos\beta\cos\alpha}} = \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha}$$

6. Calculation of the expression of the tangent of angle APB.

$$f(x) = \tan(\beta - \alpha),\ \beta = \tan^{-1}\left(\frac{b}{x}\right),\ \alpha = \tan^{-1}\left(\frac{a}{x}\right)$$

$$f(x) = \frac{\frac{b}{x} - \frac{a}{x}}{1 + \frac{b}{x} \cdot \frac{a}{x}}$$

$$f(x) = \frac{(b-a)x}{ab + x^2}$$

7. Calculation of the derivative of this expression.

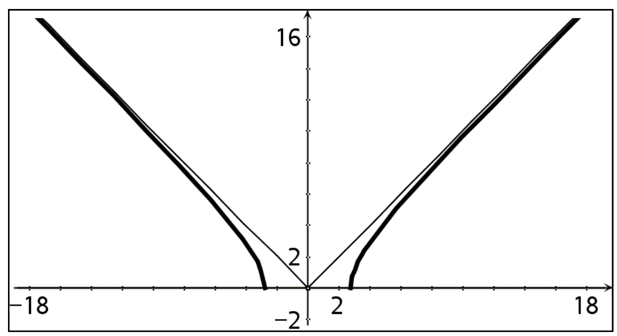
$$f'(x) = \frac{(b-a)(ab - x^2)}{(ab + x^2)^2}$$

8. Value at which the derivative equals 0.

$x_{\max} = \sqrt{ab}$ because it must be the case that $x > 0$

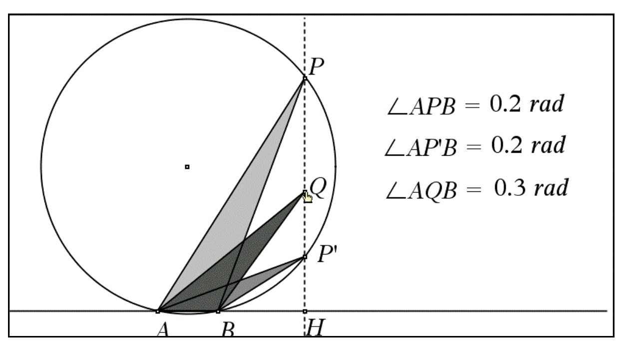
Extensions to this Problem

- Students of a higher level, who have an understanding of conics, could be asked to show that all of the points at the angle's maximal position, without taking into account the constraint of a minimal value, form part of a hyperbola, the asymptotes of which should be stated.



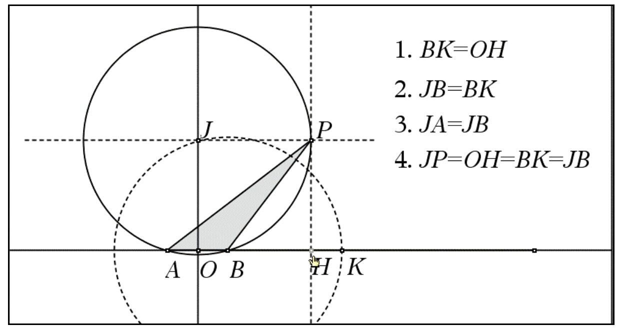
- Justify the fact that in practice right-angled segments are obtained in the diagram showing the optimal position (approximation of a function by a refined function…).

- Recreate the construction of the optimal point using only the properties of the inscribed angles, without having to use symbolic expression.
  The idea is to use the circle passing through points A, B and P.
  This circle generally crosses the perpendicular at two points P and P'. At P and P' the same angle is obtained; between the two the angle is greater than the inscribed angle (as we are inside the circle).



The optimal angle is therefore obtained when P=P', i.e. when this circle is in fact tangent to the perpendicular (HH'), which uses a purely geometric method of construction.

- Firstly, construct the circle with centre B and radius OH, then point J which is found at the intersection of this circle with the perpendicular bisector of [AB].
  The circle with centre J and radius JA will then be tangent to point P on the perpendicular passing through H and orthogonal to (AB).



Additional comments

- The TI-Nspire file which accompanies this problem contains some pro forma constructions, in order to limit the time spent on purely technical aspects.
- Where students have an understanding of the derivative of the inverse tangent function, it would be possible to use a direct derivation of the expression giving angle APB, without the need to work on the tangent of this angle. This also leads to a derivative which is expressed as a rational fraction.