Random Rectangles

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Grade level: secondary (Years 9-12)
Subject: mathematics
Time required: 90 minutes

Activity overview

What variables characterize a rectangle? What kind of relationships exists between these variables? In this activity you will explore this, examining patterns and forms using tables, graphs and equations.

Background

The problem can be traced back to the article by Tim Erickson “Connecting Data and Geometry.”[1] published in Mathematics Teacher 94 (Nov. 2001): 710–71. This is where I first learned about it. In Denmark it has also been introduced in connection with the so-called HOT approach to mathematics (Higher Order Thinking) where controlling variables and their relationships have a prominent position.

Concepts

- Variables: independent, dependent, stochastic, deterministic.
- Simple relationships between variables such as proportionality, inverse proportionality, linearity, quadratic relationships etc.
- Geometrical shapes: polygons and regions enclosed by simple curves (circles, parabolas, equiangular hyperbolas).
- Simple loci: Straight line, circle, parabola, equiangular hyperbola
- Algebraic modeling.

[1] The basis for this activity is a lesson from Key Curriculum Press. Fathom is a trademark of Key Curriculum Press.

Teacher preparation

This activity may be introduced via a classroom discussion *of which variables characterise an ordinary rectangle*. A rectangle may be drawn on the chalkboard or the dynamic form offered in the *TI-Nspire CAS* file may be used. Suggestions from the students are listed and may, for example, include base, height (and similar notions like width etc.), perimeter, area, angle, diagonal, volume etc. Some are relevant, some are duplicates (e.g. width and base), some are clearly based upon misunderstandings (e.g. volume) and can be dismissed. Others are irrelevant because they are actually constant, e.g. angle. In the end a set of variables are selected that should include at least the equivalent of base and height, the perimeter and the area. If a diagonal is proposed you may include that as well, but be aware that this will also introduce hyperbolas in the problem.

Once a list of variables has been selected the teacher can introduce the problem: We wish to vary the values of these variables to investigate possible relationships; but rather than varying them in a systematic manner in a geometric setting with a dynamical drawing of a rectangle (which would certainly also be a possibility) we will construct a collection of many rectangles – in this case 199 rectangles (if using the handheld device, or up to 999 on a computer!).

Again we could construct them in a systematic fashion but we will instead use a random number generator to construct random bases and heights. To keep the problem simple we will restrict the base and the height to the range from 0 to 1. If students are not familiar with the random generator **rand(),** this should also be introduced as well as the use of random seeds to avoid having all the students working on exactly the same numbers!

Classroom management tips

Students should work in pairs or small groups discussing their findings. Regular breaks when students share their thinking and progress with the class are useful as well as gentle hints from the teacher if the class is stuck. Depending upon the prerequisites of the class there may be common displays of fitting graphs to the shapes encountered. And if time permits an opportunity exists for them to obtain these results through algebraic modeling, The teacher may also steer the dialogue though simple examples. Algebraic modeling is non-trivial for most students so they may need to get some familiarity first with the process.

Technical prerequisites

Students should know how to:

- *construct lists in the* **Lists & Spreadsheet** *application*
- *draw scatter plots using lists in the* **Graphs & Geometry** *application*
- *draw graphs of elementary functions including dynamic parameters if necessary*
- *draw simple geometrical figures: polygons, circles etc.*

Step-by-step directions

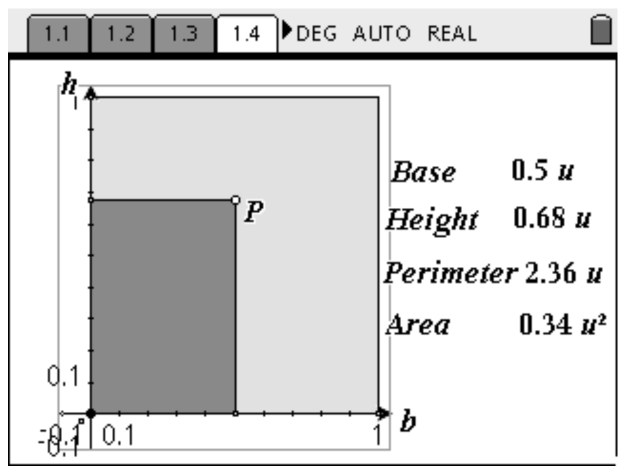
In the following we will discuss the case of four variables: **base**, **height**, **perimeter** and **area**

1. The students should classify these as **independent** and **dependent** variables. Base and height can be chosen independently. Perimeter and area then become dependent variables (or composite variables) given essentially by the sum and product of base and height:

   perimeter = 2base + 2height

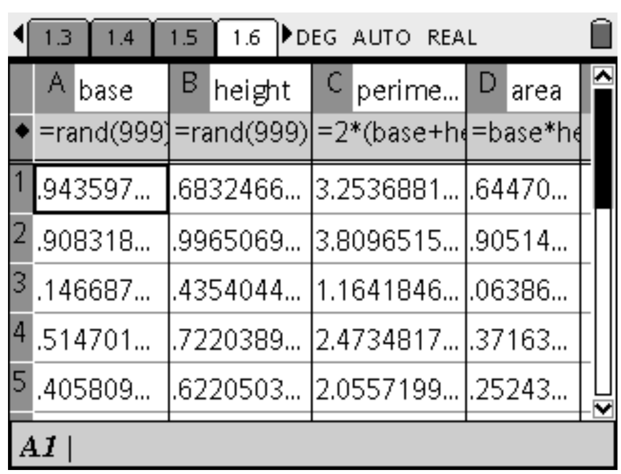
   area = base*height

   Bases and heights are entered in the **Lists & Spreadsheet** tool as lists with the appropriate names. Afterwards the generating formulas **base = rand(199), height = rand(199)** should be deleted, so that they no longer behave like stochastic variables. This emphasizes their status as independent variables (i.e. they are not generated by formulas dependent upon other variables).



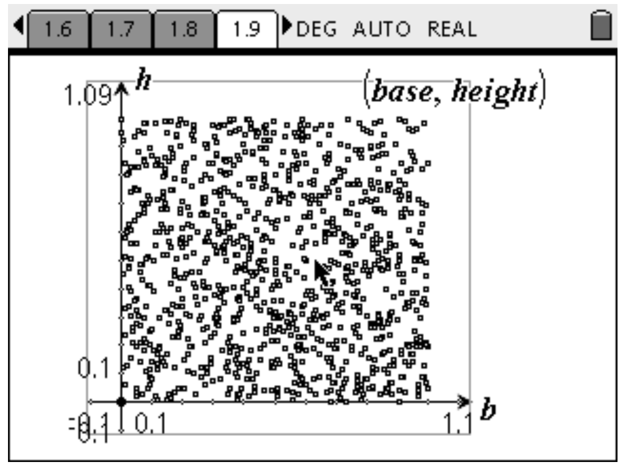
2. These lists can now be transferred pair wise to the **Graphs & Geometry** application so that we can investigate their relationship via a scatter plot.

Students should also give some thought to which combinations will carry new information. Base and height are essentially equivalent, so (base, perimeter) and (height, perimeter) generates the same relationship. This leaves four different combinations to consider:(base, height), (base, perimeter), (base, area) and (perimeter, area). *Notice in the last case, that now the perimeter is treated as an independent variable!*

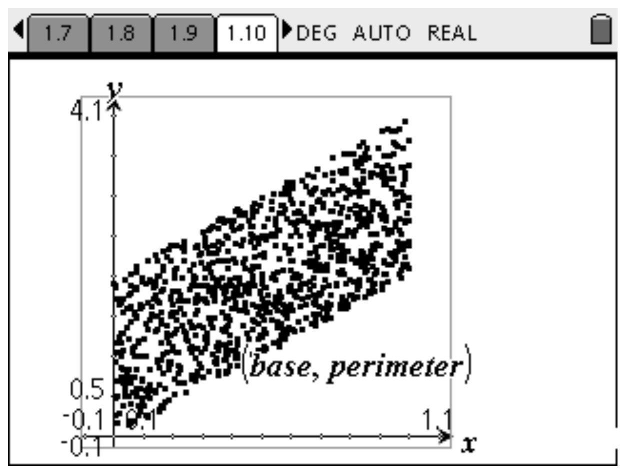


3. These four cases each gives rise to their own characteristic shape which the students should identify:

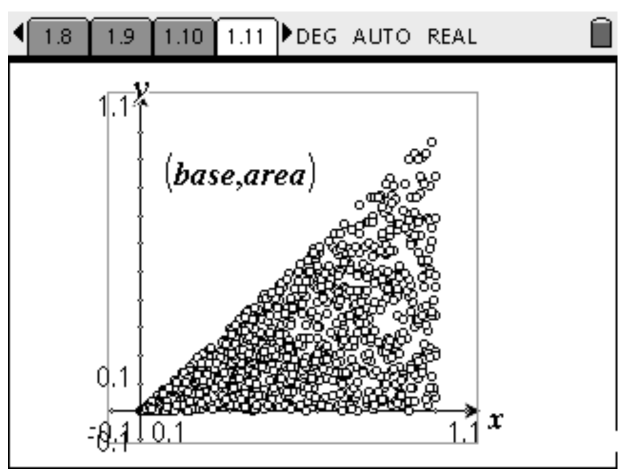
   (base, height): This generates a unit square with a uniform distribution of the points inside the unit square

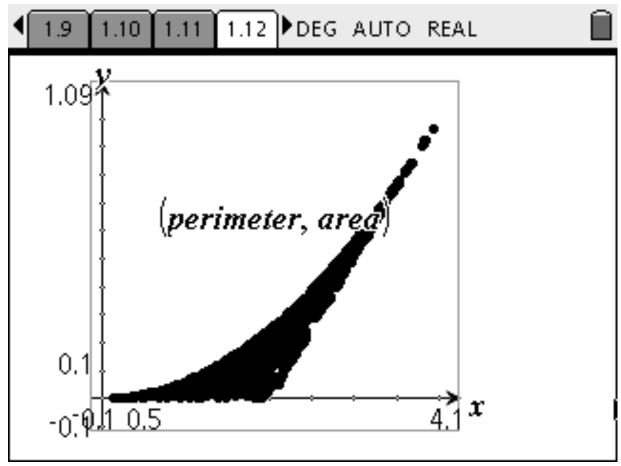


   (base, perimeter): This generates a trapezoid with a uniform distribution of points inside.

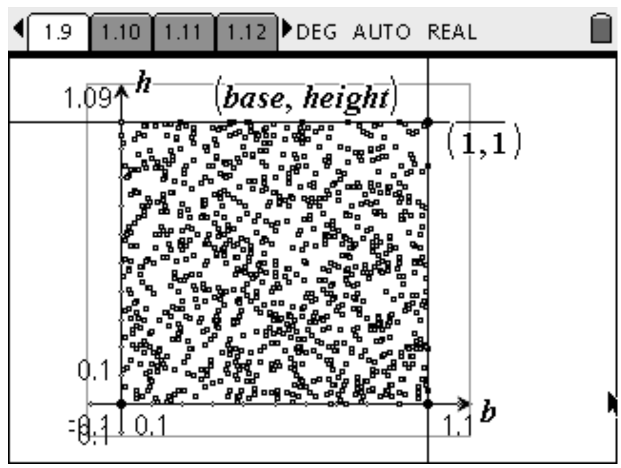


   (base, area): This generates an isosceles right-angled triangle (a half square) but this time the distribution of the points inside is not uniform!

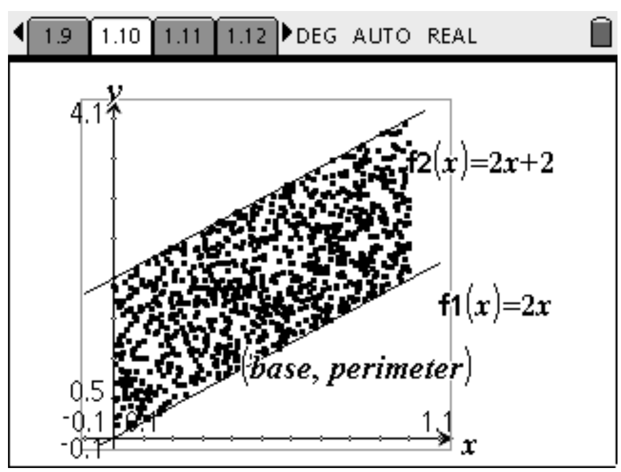




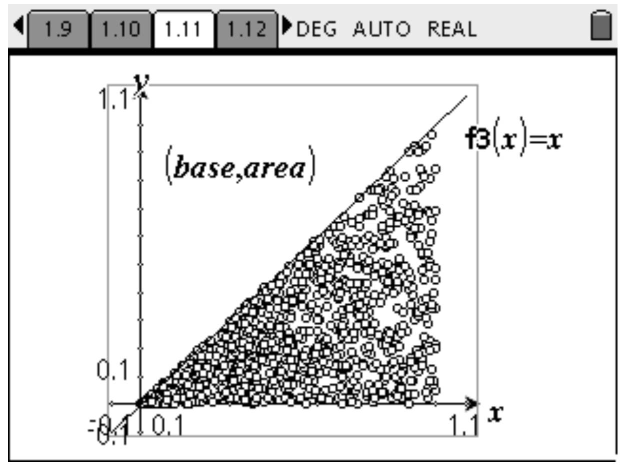
(perimeter, area): This is the most interesting shape because it's no longer polygonal. Students may guess the shape of the curvilinear boundary. Some may think it is circular, which is easy to test, some may think it's a parabolic arc.

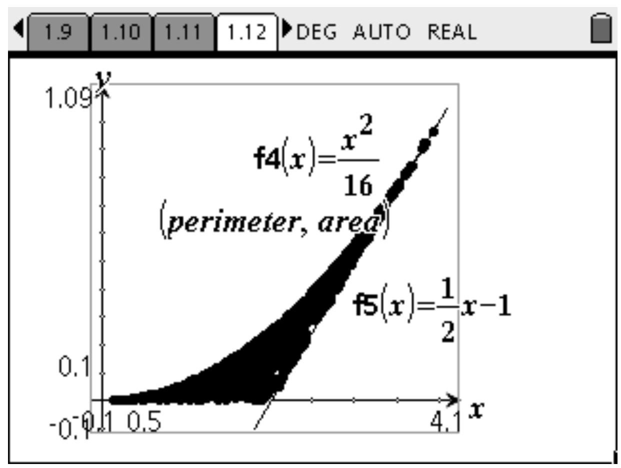


4. Students should clearly outline the shapes using the grid to position the line segments and the function grapher to position the parabolic segment. Students should be encouraged to guess the equations of the boundary lines. Subsequently they can ask for the equations using the coordinate/equation tool. As for the parabola $y = k{\cdot}x^2$ they should either try to determine the coefficient k using a dynamic parameter or they should try to guess a second point on the graph. Here the terminating point is very convenient. Students should also try to restrict the parabolic arc to its appropriate range[2].



5. If time permits students should now be encouraged to do an algebraic modeling explaining the origin of the relationships. Students should reflect on the nature of the boundary points: what are the characteristics of the corresponding rectangles?
The most difficult case is the fourth case concerning the relationship between perimeter and area, so we will take a closer look at that:


1.9
1.10
1.11
1.12
DEG AUTO REAL
1.1
y
$(base,area)$
$f3(x)=x$
0.1
-0.1
0.1
1.1
x


1.9
1.10
1.11
1.12
DEG AUTO REAL
1.09
y
$f4(x)=\frac{x^2}{16}$
$(perimeter, area)$
$f5(x)=\frac{1}{2}x-1$
0.1
-0.1
0.5
4.1
x

Assessment and evaluation

- Answers to student questions that are included in the TI-Nspire™ document
- Students should be encouraged to write a report explaining the experiment and the reasoning behind their discoveries. Student reports of this investigation should be clear and detailed; observations should be documented with appropriate screen dumps and clear explanations of each representation (tabular, geometrical, graphical and algebraic) should be required. If possible the students should try to derive the equations using algebraic modeling.

Activity extensions

- Including a diagonal will introduce further fascinating shapes that are now also bounded by rotated hyperbolas such as $y = \sqrt{x^2 + 1}$. The locus of the endpoint of a diagonal with a specific length is a circle having this length as a radius. This permits a not too complicated analysis of the relevant equations for the relationships.
- Investigating rectangular boxes in 3 dimensions using the same strategies will give the students an interesting opportunity to consolidate and generalize the exercise.
- There are also very interesting connections to probability theory: Investigating the distribution of the stochastic variables **perimeter** and **area** corresponds to investigating sums and products of independent stochastic variables and determining the density functions for these variables.